

A QUANTITATIVE DIFFERENTIAL EQUATION APPROXIMATION FOR A ROUTING MODEL

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ABSTRACT. We consider a Markovian load balancing model on a fully-connected network, where calls have Poisson arrivals and exponential durations. The endpoints of each call are uniform over all the links of the network. Each call is routed either along the link connecting its endpoints, or, if the direct route is unavailable, along a two-link path between them, via an intermediate node. We use an explicit and simple coupling to show a strong concentration of measure property, and deduce that the evolution of the process may be approximated by a differential equation. The technique is likely to be useful in other settings.

S: intro

1. INTRODUCTION

We consider a class of online routing problems in continuous time, where calls have Poisson arrivals and exponential durations, studied earlier in [1; 3; 4; 5; 7; 9].

The setting is as follows. For each $n \in \mathbb{N}$, we have a fully connected *communication graph* K_n , with node set $V_n = \{1, \dots, n\}$ and link set $E_n = \{\{u, v\} : 1 \leq u < v \leq n\}$. Each link $\{u, v\} \in E_n$ has capacity $C = C(n) < \infty$, where $C(n) \in \mathbb{Z}^+$ for each n . Each arriving call is to be routed either along a link $\{u, v\}$ or along a path between u and v consisting of a pair of links $\{u, w\}$ and $\{v, w\}$, for a pair u, v of distinct nodes (endpoints of the call) and some intermediate node $w \neq u, v$, if possible. A call in progress will use one unit of capacity of each of the links it occupies, for its entire duration. Calls arrive as a Poisson process with rate $\lambda \binom{n}{2}$, where $\lambda = \lambda(n) > 0$. The endpoints of each call are uniform over the links of the complete graph K_n . If a call is for nodes u and v , then we route it over the direct link $\{u, v\}$ between u and v if possible, that is if $\{u, v\}$ has fewer than C calls currently using it. Otherwise, we pick an ordered list of $d = d(n)$ possible intermediate nodes (w_1, \dots, w_d) from $V_n \setminus \{u, v\}$, uniformly at random with replacement, and the call is routed along one of the two-link routes $\{\{u, w_1\}, \{v, w_1\}\}, \dots, \{\{u, w_d\}, \{v, w_d\}\}$, chosen to minimise the larger of the current loads on its two links, subject to the

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capacity constraints. Ties are broken in favour of the first ‘best’ route in the ordered list. If none of the d two-link paths is available, then the call is lost. Call durations are unit mean exponential random variables, independent of one another and of the arrivals and choices processes.

Here, we focus on the analysis of this algorithm as n tends to infinity. We prove that, asymptotically, for suitable initial conditions and suitable functions $\lambda(n)$, $d(n)$ and $C(n)$, for each node v , the proportion of links at v that carry k calls is well approximated by the solution of a differential equation. (Note that, when λ , d and C vary with n , there is no single limiting differential equation, but rather a sequence of approximating differential equations, with dimension tending to infinity if $C \rightarrow \infty$ with n .)

Such law of large numbers results have been difficult to prove in this and related models, due to apparent and potentially strong dependencies between system elements (in this context, links). Here we are able to prove that these dependencies are negligible; it turns out that, in a suitable sense, links in certain collections evolve approximately independently of one another. Our technique appears to be new, and is likely to be useful in other settings. It relies on a coupling, which is used to prove that slowly-changing functions of the process (for instance, the number of links around a node v with load exactly k , for each node v and each $k \in \{0, 1, \dots, C\}$) are well concentrated at each time t . Thanks to the strong concentration of measure, it is then possible to show that the expected drifts of functions of interest factorise approximately, leading to a differential equation approximating these functions. The basic principle of our approach is, in essence, rather simple; however, there are considerable complications arising from the complicated evolution of the process in question.

For each link $e = \{u, v\} \in E_n$, let $X_t^{(n)}(e, 0)$ denote the number of calls in progress at time t which are routed along the link e , that is the number of directly routed calls between the end nodes u and v of e that are in progress at time t . For each link $e = \{u, v\} \in E_n$ and node $w \in V_n \setminus e$, let $X_t^{(n)}(e, w)$ denote the number of calls in progress at time t which are routed along the path consisting of links $\{u, w\}, \{v, w\}$, that is the number of calls between the end nodes u and v of e routed via w that are in progress at time t . We call $X_t^{(n)} = (X_t^{(n)}(e, 0), X_t^{(n)}(e, w) : e \in E_n, w \in V_n \setminus e)$ the *load vector* at time t , and let $S = \{0, 1, \dots, C\}^{n(n-1)^2/2}$ denote the state space, containing the set of all possible load vectors. Then $X^{(n)} = (X_t^{(n)})_{t \geq 0}$ is a continuous-time discrete-space Markov chain. We will normally drop the superscript n , to avoid unnecessarily cluttering the notation.

Let $\tilde{S} \subseteq S$ be the set of load vectors x such that

$$\|x\|_1 = \sum_e x(e, 0) + \sum_{e, w \notin e} x(e, w) \leq 6\lambda \binom{n}{2},$$

that is the subset of the state space consisting of those load vectors where the total number of calls in the network is at most $6\lambda \binom{n}{2}$. Let T be the

first time t such that $X_t \notin \tilde{S}$. Then T is a stopping time with respect to the natural filtration of process X . Let X^T be the Markov process given as follows: $X_t^T = X_t$ for $t < T$, and $X_t^T = X_T$ for $t \geq T$. In what follows, we omit the superscript T , again to lighten notation, and write simply X , but it is to be understood that we only consider the evolution of the process until it leaves \tilde{S} . In the case where the link capacity C is constant, it would be slightly simpler to proceed without restricting the evolution of X to \tilde{S} , but we will not bother treating this case differently.

Given a load vector $x \in S$ and a link $e = \{u, v\} \in E_n$, let $x(e)$ denote the load of link e . Then

$$x(\{u, v\}) = x(\{u, v\}, 0) + \sum_{w \notin \{u, v\}} (x(\{u, w\}, v) + x(\{v, w\}, u)).$$

Given a load vector x , node v and $k \in \mathbb{Z}^+$, let $f_{v,k}(x)$ be the number of links $\{v, w\}$, $w \neq v$, in x such that $x(\{v, w\}) = k$ (that is, the number of links with one end v carrying exactly k calls).

For a vector $\xi = (\xi(k) : k = 0, \dots, C)$, let

$$\xi(\leq j) = \sum_{k=0}^j \xi(k).$$

Define $F : \mathbb{R}^{C+1} \rightarrow \mathbb{R}^{C+1}$ by

$$\begin{aligned} F_0(\xi) &= -\lambda\xi(0) - \lambda g_0(\xi) + \xi(1), \\ F_k(\xi) &= \lambda\xi(k-1) - \lambda\xi(k) + \lambda g_{k-1}(\xi) - \lambda g_k(\xi) - k\xi(k) \\ &\quad + (k+1)\xi(k+1), \quad 0 < k < C, \\ F_C(\xi) &= \lambda\xi(C-1) + \lambda g_{C-1}(\xi) - C\xi(C), \end{aligned} \tag{1.1}$$

where functions g_j , $j = 0, \dots, C-1$, are given by

$$\begin{aligned} g_j(\xi) &= 2\xi(C)\xi(j)\xi(\leq j) \sum_{r=1}^d (1 - \xi(\leq j)^2)^{r-1} (1 - \xi(\leq j-1)^2)^{d-r} \\ &\quad + 2\xi(C)\xi(j) \sum_{i=j+1}^{C-1} \xi(i) \sum_{r=1}^d (1 - \xi(\leq i)^2)^{r-1} (1 - \xi(\leq i-1)^2)^{d-r} \\ &= 2\xi(C)\xi(j)\xi(\leq j) \frac{(1 - \xi(\leq j-1)^2)^d - (1 - \xi(\leq j)^2)^d}{\xi(\leq j)^2 - \xi(\leq j-1)^2} \\ &\quad + 2\xi(C)\xi(j) \sum_{i=j+1}^{C-1} \xi(i) \frac{(1 - \xi(\leq i-1)^2)^d - (1 - \xi(\leq i)^2)^d}{\xi(\leq i)^2 - \xi(\leq i-1)^2}. \end{aligned} \tag{1.2}$$

When $d = 1$, $g_j(\xi) = 2\xi(C)(1 - \xi(C))\xi(j)$ for $0 \leq j \leq C-1$. When $d = 2$, for $0 \leq j \leq C-1$,

$$g_j(\xi) = 2\xi(C)\xi(j)\xi(\leq j)(2 - (\xi(\leq j))^2 - (\xi(\leq j-1))^2)$$

$$+ 2\xi(C)\xi(j)\left(\sum_{j < i < C} \xi(i)(2 - (\xi(\leq i))^2 - (\xi(\leq i-1))^2)\right).$$

We will show (see Lemma 7.1) that, for each $d \geq 1$, the function $F(\xi)$ is Lipschitz, with constant $8d^2(\lambda + 1)(C + 1)^2$, over the set of non-negative ξ such that $\sum_{k=0}^C \xi(k) \leq 1$, with respect to the ℓ_∞ norm. Hence we will prove that the equation

$$\frac{d\xi_t}{dt} = F(\xi_t) \tag{1.3} \quad \boxed{\text{eq.diff-eq}}$$

has a unique solution starting from ξ_0 such that $\xi_0(j) \geq 0$ for all j and $\sum_{j=0}^C \xi_0(j) = 1$, valid for all times and such that, for all t , $\xi_t(j) \geq 0$ for each j and $\sum_{j=0}^C \xi_t(j) = 1$.

In equation (1.3), the linear terms account for the effects of calls routed directly, and for call departures. Each function g_j is proportional to the rate of arrivals of alternatively routed calls onto links that carry j calls.

Given a pair of nodes u, v and $j \in \{0, \dots, C\}$, let $\mathbb{I}_{uv}^j : S \rightarrow \{0, 1\}$ be defined by $\mathbb{I}_{uv}^j(x) = 1$ if $x(\{u, v\}) = j$ and $\mathbb{I}_{uv}^j(x) = 0$ otherwise. Thus \mathbb{I}_{uv}^j is the indicator of the set of load vectors where the load of link $\{u, v\}$ is j . Note that $\mathbb{I}_{uv}^j = \mathbb{I}_{vu}^j$, and that \mathbb{I}_{vv}^j is identically 0 for each v and j .

Let S_1 be the set of all states x such that $\|x\|_1 \leq 2\lambda \binom{n}{2}$.

Let functions $\phi^1, \phi^2, \phi^3 : S \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} \phi^1(x) &= \max_{u,v:u \neq v} \max_{j,k} \left| \frac{1}{n-2} \sum_w \mathbb{I}_{uw}^j(x) \mathbb{I}_{vw}^k(x) \right. \\ &\quad \left. - \frac{1}{(n-2)^2} \sum_{w \neq u,v} \mathbb{I}_{uw}^j(x) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(x) \right|; \end{aligned} \tag{1.4}$$

$$\begin{aligned} \phi^2(x) &= \max_{u,v:u \neq v} \max_j \frac{1}{n-2} |f_{u,j}(x) - f_{v,j}(x)| \\ &= \max_{u,v:u \neq v} \max_j \frac{1}{n-2} \left| \sum_{w \neq u} \mathbb{I}_{uw}^j(x) - \sum_{w \neq v} \mathbb{I}_{vw}^j(x) \right| \\ &= \max_{u,v:u \neq v} \max_j \frac{1}{n-2} \left| \sum_{w \neq u,v} \mathbb{I}_{uw}^j(x) - \sum_{w \neq u,v} \mathbb{I}_{vw}^j(x) \right|; \end{aligned} \tag{1.5}$$

$$\phi^3(x) = \max_{u,v:u \neq v} \frac{1}{n-2} \sum_{w \neq u,v} x(\{u, v\}, w). \tag{1.6}$$

Let $\phi = \max\{\phi^1, \phi^2, \phi^3\}$. This function ϕ is related to the function called ϕ by Crametz and Hunt in [3].

We shall prove that, if $\phi(X_0)$ is small, for instance $\phi(X_0) = O\left(\frac{\log n}{\sqrt{n}}\right)$, then $\phi(X_t)$ remains small for a time interval of order 1, and over that period each function $(n-1)^{-1}f_{v,j}(X_t)$ is well-approximated by the solution to the differential equation (1.3) with initial condition $\xi_0(j) = (n-1)^{-1}f_{v,j}(X_0)$ for

$j = 0, \dots, C$. Note that, if $\phi^2(X_0)$ is small, then, for each j , all the functions $f_{v,j}(X_0)$ for different $v \in V_n$ are nearly equal, so all the $(n-1)^{-1}f_{v,j}(X_t)$ can be approximated by the same solution to the differential equation (1.3).

thm.main-result

Theorem 1.1. *Suppose that λ and t_0 are positive reals, and d and C positive integers. Set $\gamma = 1/(2^{24}d^8(8\lambda t_0 + 1)^3 e^{800d\lambda t_0})$, and suppose that*

$$n \geq \max\left(\frac{1}{\lambda}, \frac{1}{8\lambda t_0}, 10^7(\lambda + 1)^4 d^4 (C + 1)^6 t_0^2, e^{8/\gamma}\right).$$

Let ξ_0 satisfy $\xi_0(j) \geq 0$ for all $j = 0, \dots, C$, and $\sum_{j=0}^C \xi_0(j) = 1$. Let (ξ_t) be the unique solution to the differential equation (1.3) on $[0, t_0]$, subject to initial condition ξ_0 . Let X_0 be in S_1 . Let A_n be the event that, for each $v \in V_n$, each $k \in \{0, \dots, C\}$ and each $t \in [0, t_0]$,

$$\begin{aligned} |f_{v,k}(X_t) - (n-1)\xi_t(k)| &\leq \left(\sup_{u,j} |f_{u,j}(X_0) - (n-1)\xi_0(j)|\right. \\ &\quad \left.+ 46(\lambda + 1)(t_0 + 1)d^2(C + 1)^3 \left(n\phi(X_0) + 3n^{1/2} \log n\right)\right) e^{216(\lambda+1)d^2(C+1)^3 t_0}. \end{aligned}$$

Then $\mathbb{P}(\overline{A_n}) \leq e^{-\frac{1}{2}\gamma \log^2 n}$.

In particular, suppose that there is $v_0 \in V_n$ such that $\xi_0(j) = \frac{1}{n-1}f_{v_0,j}(X_0)$ for $j = 0, \dots, C$. Let A'_n be the event that, for each $t \leq t_0$, each $k \in \{0, \dots, C\}$, and each $v \in V_n$,

$$\begin{aligned} |f_{v,k}(X_t) - (n-1)\xi_t(k)| &\leq 46(\lambda + 1)(t_0 + 1)d^2(C + 1)^3 \\ &\quad \times \left(2n\phi(X_0) + 3n^{1/2} \log n\right) e^{216(\lambda+1)d^2(C+1)^3 t_0}. \end{aligned}$$

Then $\mathbb{P}(\overline{A'_n}) \leq e^{-\frac{1}{2}\gamma \log^2 n}$.

For the special case $d = 1$, we obtain sharper bounds, replacing the term $(C + 1)^3$ in the exponent by $(C + 1)$.

thm.main-result-d=1

Theorem 1.2. *Suppose that λ and t_0 are positive reals, and C a positive integer, and suppose that $d = 1$. Set $\gamma = 1/(2^{24}(8\lambda t_0 + 1)^3 e^{800\lambda t_0})$, and suppose that*

$$n \geq \max\left(\frac{1}{\lambda}, \frac{1}{8\lambda t_0}, 10^7(\lambda + 1)^4 (C + 1)^6 t_0^2, e^{8/\gamma}\right).$$

Let ξ_0 satisfy $\xi_0(j) \geq 0$ for all $j = 0, \dots, C$, and $\sum_{j=0}^C \xi_0(j) = 1$. Let (ξ_t) be the unique solution to the differential equation (1.3) on $[0, t_0]$, subject to initial condition ξ_0 . Let X_0 be in S_1 . Let B_n be the event that, for each $v \in V_n$, each $k \in \{0, \dots, C\}$ and each $t \in [0, t_0]$,

$$\begin{aligned} |f_{v,k}(X_t) - (n-1)\xi_t(k)| &\leq \sup_{u,j} |f_{u,j}(X_0) - (n-1)\xi_0(j)| \\ &\quad + 46(\lambda + 1)(t_0 + 1)(C + 1)^3 \left(n\phi(X_0) + 3n^{1/2} \log n\right) e^{216(\lambda+1)(C+1)t_0}. \end{aligned}$$

Then $\mathbb{P}(\overline{B_n}) \leq e^{-\frac{1}{2}\gamma \log^2 n}$.

Suppose that, for each n , $X_0^{(n)} = x_0^{(n)}$ a.s. for some deterministic load vector $x_0^{(n)}$ such that, for some constant c , $\phi(x_0^{(n)}) \leq \frac{c \log n}{\sqrt{n}}$ and $\max_{v,j} |(n-1)^{-1} f_{v,j}(x_0^{(n)}) - \xi_0(j)| \leq \frac{c \log n}{\sqrt{n}}$. Suppose also that, as $n \rightarrow \infty$, λ and t_0 are bounded away from 0, and that $\lambda d^2 C^3 t_0 = o(\log n)$ and $d \lambda t_0 = o(\log \log n)$. Then, for sufficiently large n , the condition on n in Theorem 1.1 is satisfied, and the theorem implies that, for $\epsilon > 0$, if A_n^ϵ is the event that, for each $v \in V_n$, each $k \in \{0, \dots, C\}$, and each $t \in [0, t_0]$,

$$|f_{v,k}(X_t) - (n-1)\xi_t(k)| \leq n^{1/2+\epsilon},$$

then $\mathbb{P}(\overline{A_n^\epsilon}) \rightarrow 0$ as $n \rightarrow \infty$. For $d = 1$, the corresponding conditions are that, as $n \rightarrow \infty$, λ and t_0 are bounded away from 0, and that $\lambda C t_0 = o(\log n)$ and $\lambda t_0 = o(\log \log n)$.

In the simplest case where $d = 1$, if the direct link is at full capacity, only one two-link alternative route is considered, and it is used if there is spare capacity on both links. This case, with constant arrival rate λ and constant capacity C , was first studied by Gibbens, Hunt and Kelly [4] and then by Crametz and Hunt [3], and Graham and Méléard [5]. For $k = 0, 1, \dots, C$, let $Y_t^{(n)}(k)$ denote the proportion of links that carry k calls at time t in a system with n nodes. It is conjectured in [4] and shown in [3] that, under suitable conditions, $(Y_t^{(n)}(k) : k = 0, \dots, C)$ converges in distribution as $n \rightarrow \infty$ to a deterministic vector $(Y_t(k) : k = 0, \dots, C)$ obtained as the solution to the differential equation derived from the average drift of $(Y_t^{(n)}(k) : k = 0, \dots, C)$, with appropriate initial conditions. The convergence result in [3] is non-quantitative. Graham and Méléard [5] do give a quantitative result concerning independence of small collections of links under the assumptions that initial link loads are iid and that initially there are no alternatively routed calls in the network. This result can be used to deduce a quantitative law of large numbers. (Also, it would be possible to quantify convergence in the more general case they consider.)

In the case of λ and C constant, and $d = 1$, Theorem 1.2 is a more refined, quantitative, version of the law of large numbers in [3]. Also, our result in this case is related to those in [5]. Unlike [5], we do not need to assume that initially all the nodes are exactly exchangeable. Instead, our law of large numbers result holds for a large class of deterministic initial states, and holds simultaneously for all nodes. Theorem 1.1 and the remaining cases in Theorem 1.2 are completely new.

For $d \geq 2$ constant and constant λ , this model is a variant of one that has attracted earlier interest. Luczak and Upfal [9] study a version (both in discrete and in continuous time) where the total capacity of each link $\{u, v\}$ is divided into three parts, one for ‘direct’ calls, one for indirectly routed calls with one end u and one for indirectly routed calls with one end v . Equivalently, each ‘undirected’ link $\{u, v\}$ has capacity $C_1(n)$ and is a first-choice path for calls between u and v ; also, for each link $\{u, v\}$ there

are two directed links, uv and vu , each with capacity $C_2(n)$. Link uv is used for indirectly routed calls with one end u and link vu is used for indirectly routed calls with one end v . The results of [9] for the discrete-time model were strengthened and extended by Luczak, McDiarmid and Upfal [8], who also studied the discrete-time version of the model that is the focus of this paper. The long-term behaviour of the continuous-time model was analysed in [1] and also in [7], where calls are not routed on direct links at all.

Theorem 1.1 holds also in the case above where direct links are not used (i.e., each arriving call is allocated to the best among d indirect routes), with a suitably modified function F in the differential equation (1.3). Indeed, for $0 < k < C$, we take instead

$$F_k(\xi) = \lambda g_{k-1}(\xi) - \lambda g_k(\xi) - k\xi(k) + (k+1)\xi(k+1),$$

where the functions $g_k(\xi)$ are amended by dropping the factor $\xi(C)$; $F_0(\xi)$ and $F_C(\xi)$ are modified in the same way. The proof is essentially identical, indeed slightly simpler in a few places.

In [7] (as well as in [8] for a corresponding discrete-time model), the class of routing strategies choosing a path for a new call from among d random alternatives is called the GDAR (General Dynamic Alternative Routing) Algorithm. The particular model we study in this paper is called the BDAR (Balanced Dynamic Alternative Routing) Algorithm. The FDAR (First Dynamic Alternative Routing) Algorithm always chooses the first possible alternative two-link route among the d chosen. As in other models of this type, the ‘power of two choices’ phenomenon has been observed, that is, with the BDAR algorithm for $d \geq 2$, the capacity $C = C(n)$ required to ensure that most calls are routed successfully is much smaller than with the FDAR algorithm. This phenomenon is not exhibited by the model studied in this paper, as proved in Theorem 1.1 of [8], but it does occur in the variant discussed above where direct links are not used.

In particular, see [7], for the variant where there is capacity division into three parts, but with zero capacity for direct calls, the following is true after a ‘burn-in’ period of length $O(\log n)$. Suppose we use the FDAR algorithm and each indirect link has capacity $C(n) \sim \alpha \frac{\log n}{\log \log n}$, where $\alpha > 0$ is a constant. If $\alpha > 2/d$, then there exists a constant $K > 0$ such that the mean number of calls lost in an interval of length n^K is $o(1)$. If $\alpha < 2/d$ then for each K there exists a constant $c > 0$ such that the mean number of calls lost in an interval of length n^K is at least n^c . On the other hand, suppose we use the BDAR algorithm with $d \geq 2$ choices, and let $K > 0$ be a constant. There exists a constant $c > 0$ such that, if $C(n) \geq \frac{\log \log n}{\log d} + c$ then the expected number of lost calls in an interval of length n^K is $o(1)$; and if $C(n) \leq \frac{\log \log n}{\log d} - c$ then the expected number of lost calls in such an interval is at least $n^{K+2-o(1)}$ as $n \rightarrow \infty$.

Our methods apply to any GDAR algorithm, and indeed any of the variants discussed above. The law of large numbers for the BDAR algorithm

proved here is valid for the model without direct links in the parameter range considered in [7], i.e., with constant λ and d , and $C = C(n) = O(\log \log n)$ for $d \geq 2$, and $C = C(n) = O(\log n / \log \log n)$ for $d = 1$.

In [2], we carry out an improved analysis of the coupling introduced in this paper, to prove rapid convergence to equilibrium, and concentration of measure in equilibrium, for small enough arrival rate λ . Also, for such λ , we are able to show that $\frac{1}{n-1}f_{v,k}$ is well-approximated by the unique fixed point of the differential equation (1.3). For $d = 1$, this proves the approximation suggested in [4], and it also provides an alternative proof of some of the results in [7], for small enough λ .

The rest of the paper is organised as follows. In Section 2 we develop a concentration of measure inequality that will be a fundamental ingredient of our proofs. In Section 3, we formally write down the generator of the Markov chain in question, and give an informal explanation of our proof strategy. In Section 4, we describe a simple coupling between two copies of the Markov chain, and show that, under the coupling, they do not get much further apart over time, according to suitable notions of distance. In Section 5, we use the coupling to show that nice functions of the Markov chain are well concentrated around their expectations. Section 6 contains estimates of the expectation of the generator of the Markov chain. In Section 7, we prove our main results, Theorems 1.1 and 1.2. In Section 8, we discuss the issue of initial conditions needed for Theorem 1.1 and 1.2 to apply. In Section 9 we discuss how our techniques can be extended to analyse other models.

2. CONCENTRATION INEQUALITIES

S: conc

We present some concentration of measure inequalities that will be used in our proofs, and may also be useful in the analysis of other Markov chains with similar properties. These inequalities generalise results presented in [6].

Let $X = (X_t)_{t \in \mathbb{Z}^+}$ be a discrete-time Markov chain with a discrete state space S and transition probabilities $P(x, y)$ for $x, y \in S$, where $\sum_{y \in S} P(x, y) = 1$ for each $x \in S$. We allow X to be lazy, that is we allow $P(x, x) > 0$ for some $x \in S$. We assume that, for each $x \in S$, $|\{y : P(x, y) > 0\}| < \infty$.

This setting is natural, and many models in applied probability and combinatorics fit into this framework, including those discussed in Section 1.

Let

$$\Omega = S^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \dots) : \omega_i \in S \quad \forall i\}.$$

Members ω of Ω will correspond to possible paths of the chain X , in that $X_i(\omega) = \omega_i$ for $i \in \mathbb{Z}^+$. Then for each $t \in \mathbb{Z}^+$, X_t may be viewed as a random variable on a measurable space (Ω, \mathcal{F}) , where $\mathcal{F} = \sigma(\cup_{t=0}^{\infty} \mathcal{F}_t)$ and $\mathcal{F}_t = \sigma(X_i : i \leq t)$. The σ -fields \mathcal{F}_t form the natural filtration for X .

Let $\mathcal{P}(S)$ be the power set of the discrete set S . The law of the Markov chain can be interpreted as a probability measure \mathbb{P} on (Ω, \mathcal{F}) , and is determined uniquely by the transition matrix P together with the initial state

$X_0 = x_0$, according to

$$\mathbb{P}(\{\omega : \omega_j = x_j \text{ for all } j \leq i\}) = \prod_{j=0}^{i-1} P(x_j, x_{j+1}),$$

for each $x_1, \dots, x_i \in S$, for each $i \in \mathbb{Z}^+$. To be precise, this defines the law of (X_t) conditional on $X_0 = x_0$, and will be denoted by \mathbb{P}_{x_0} in what follows. Let $P^t(x, y)$ be the t -step transition probability from x to y , given inductively by

$$P^t(x, y) = \sum_{z \in S} P^{t-1}(x, z)P(z, y).$$

Then, for $A \subseteq S$,

$$\mathbb{P}_{x_0}(X_t \in A) = \mathbb{P}_{x_0}(\{\omega : \omega_t \in A\}) = (\delta_{x_0} P^t)(A).$$

Let \mathbb{E}_{x_0} denote the expectation operator corresponding to \mathbb{P}_{x_0} ; then $\mathbb{E}_{x_0}[f(X_t)]$ is the expectation of the function f with respect to measure $\delta_{x_0} P^t$.

For $t \in \mathbb{Z}^+$ and $f : S \rightarrow \mathbb{R}$, define the function $P^t f$ by

$$(P^t f)(x) = \sum_y P^t(x, y) f(y), \quad x \in S.$$

Thus $(P^t f)(x) = \mathbb{E}_x[f(X_t)]$ for each x . Also, for $x \in S$, let $N(x) = \{y \in S : P(x, y) > 0\}$.

The following concentration of measure result for real-valued functions of X_t is presented to set the scene, and because it may prove to be of independent interest. We will not use it in the proof of Theorems 1.1 and 1.2.

thm.conca-general

Theorem 2.1. *Let P be the transition matrix of a discrete-time Markov chain with discrete state space S . Let $f : S \rightarrow \mathbb{R}$ be a function.*

(i) *Let $(\alpha_i)_{i \in \mathbb{Z}^+}$ be a sequence of positive constants such that for all $i \in \mathbb{Z}$,*

$$\sup_{x \in S, y \in N(x)} |\mathbb{E}_x[f(X_i)] - \mathbb{E}_y[f(X_i)]| \leq \alpha_i. \quad (2.1) \quad \text{cond-gen}$$

Then for all $a > 0$, $x_0 \in S$, and $t > 0$,

$$\mathbb{P}_{x_0}(|f(X_t) - \mathbb{E}_{x_0}[f(X_t)]| \geq a) \leq 2e^{-a^2/2(\sum_{i=0}^{t-1} \alpha_i^2)}. \quad (2.2) \quad \text{ineq.conca-general}$$

(ii) *More generally, let S_0 be a non-empty subset of S , and let $(\alpha_i)_{i \in \mathbb{Z}^+}$ be a sequence of positive constants such that, for all $i \in \mathbb{Z}$,*

$$\sup_{x, y \in S_0 : y \in N(x)} |\mathbb{E}_x[f(X_i)] - \mathbb{E}_y[f(X_i)]| \leq \alpha_i. \quad (2.3) \quad \text{cond-gen-1}$$

Let $S_0^0 = \{x \in S_0 : y \in S_0 \text{ whenever } P(x, y) > 0\}$. Then for all $x_0 \in S_0^0$, $a > 0$ and $t > 0$,

$$\mathbb{P}_{x_0} \left(\{|f(X_t) - \mathbb{E}_{x_0}[f(X_t)]| \geq a\} \cap \{X_s \in S_0^0 : 0 \leq s \leq t-1\} \right) \leq 2e^{-a^2/2(\sum_{i=0}^{t-1} \alpha_i^2)}. \quad (2.4) \quad \text{ineq.conca}$$

Our proof of Theorem 2.1 makes use of a concentration inequality from [10]. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space, with $\tilde{\Omega}$ finite. (The arguments used here could be extended to many cases where $\tilde{\Omega}$ is countably infinite.) Let $\tilde{\mathcal{G}} \subseteq \tilde{\mathcal{F}}$ be a σ -field of subsets of $\tilde{\Omega}$. Then there exist disjoint sets $\tilde{G}_1, \dots, \tilde{G}_m$ such that $\tilde{\Omega} = \cup_{r=1}^m \tilde{G}_r$ and every set in $\tilde{\mathcal{G}}$ can be written as a union of some of the sets \tilde{G}_r . Given a bounded random variable Z on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, the *conditional supremum* $\sup(Z \mid \tilde{\mathcal{G}})$ of Z in $\tilde{\mathcal{G}}$ is the $\tilde{\mathcal{G}}$ -measurable function given by

$$\sup(Z \mid \tilde{\mathcal{G}})(\tilde{\omega}) = \min_{\tilde{A} \in \tilde{\mathcal{G}}: \tilde{\omega} \in \tilde{A}} \max_{\tilde{\omega}' \in \tilde{A}} Z(\tilde{\omega}') = \max_{\tilde{\omega}' \in \tilde{G}_r} Z(\tilde{\omega}'), \quad (2.5) \quad \boxed{\text{defn-sup}}$$

where $\tilde{\omega} \in \tilde{G}_r$. Thus $\sup(Z \mid \tilde{\mathcal{G}})$ takes the value at $\tilde{\omega}$ equal to the maximum value of Z over the event \tilde{G}_r in $\tilde{\mathcal{G}}$ containing $\tilde{\omega}$.

The *conditional range* $\text{ran}(Z)$ of Z in $\tilde{\mathcal{G}}$ is the $\tilde{\mathcal{G}}$ -measurable function

$$\text{ran}(Z \mid \tilde{\mathcal{G}}) = \sup(Z \mid \tilde{\mathcal{G}}) + \sup(-Z \mid \tilde{\mathcal{G}}), \quad (2.6) \quad \boxed{\text{defn-cond-range}}$$

that is

$$\text{ran}(Z \mid \tilde{\mathcal{G}})(\tilde{\omega}) = \max_{\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{G}_r} |Z(\tilde{\omega}_1) - Z(\tilde{\omega}_2)|,$$

where $\tilde{\omega} \in \tilde{G}_r$. Note that $\sup(Z \mid \tilde{\mathcal{G}})$ and $\text{ran}(Z \mid \tilde{\mathcal{G}})$ are $\tilde{\mathcal{G}}$ -measurable.

Let $t \in \mathbb{N}$, let $\{\emptyset, \tilde{\Omega}\} = \tilde{\mathcal{F}}_0 \subseteq \tilde{\mathcal{F}}_1 \subseteq \dots \subseteq \tilde{\mathcal{F}}_t$ be a filtration in $\tilde{\mathcal{F}}$, and let Z_0, \dots, Z_t be the martingale defined by $Z_i = \tilde{\mathbb{E}}(Z \mid \tilde{\mathcal{F}}_i)$ for each $i = 0, \dots, t$. For each i , let ran_i denote $\text{ran}(Z_i \mid \tilde{\mathcal{F}}_{i-1})$; by definition, ran_i is an $\tilde{\mathcal{F}}_{i-1}$ -measurable function. For each j , let the *sum of squared conditional ranges* R_j^2 be the random variable $\sum_{i=1}^j \text{ran}_i^2$, and let the *maximum sum of squared conditional ranges* \hat{r}_j^2 be the supremum of the random variable R_j^2 , that is

$$\hat{r}_j^2 = \sup_{\tilde{\omega} \in \tilde{\Omega}} R_j^2(\tilde{\omega}).$$

The following result is Theorem 3.14 in [10].

thm.mart

Lemma 2.2. *Let Z be a bounded random variable on a finite probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\tilde{\mathbb{E}}(Z) = m$. Let $\{\emptyset, \tilde{\Omega}\} = \tilde{\mathcal{F}}_0 \subseteq \tilde{\mathcal{F}}_1 \subseteq \dots \subseteq \tilde{\mathcal{F}}_t$ be a filtration in $\tilde{\mathcal{F}}$, and assume that Z is $\tilde{\mathcal{F}}_t$ -measurable. Then for any $a \geq 0$,*

$$\tilde{\mathbb{P}}(|Z - m| \geq a) \leq 2e^{-2a^2/\hat{r}_t^2}.$$

More generally, for any $a \geq 0$ and any value r_t^2 ,

$$\tilde{\mathbb{P}}(\{|Z - m| \geq a\} \cap \{R_t^2 \leq r_t^2\}) \leq 2e^{-2a^2/r_t^2}.$$

Proof of Theorem 2.1. Let us start with (i). Let $f : S \rightarrow \mathbb{R}$ be a function. Fix a time $t \in \mathbb{N}$, and an initial state $x_0 \in S$ and consider the evolution of X_t conditional on $X_0 = x_0$ for t steps, that is until time t . Since we have assumed that there are only a finite number of possible transitions from any given $x \in S$, we can build this process until time t on a finite probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_{x_0})$: we can take $\tilde{\Omega}$ to be the finite set of all possible paths x_0, \dots, x_t of the process starting at time 0 in state x_0 until time t .

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For each time $j = 0, \dots, t$, let $\tilde{\mathcal{F}}_j = \sigma(X_0, \dots, X_j)$, the σ -field generated by X_0, \dots, X_j ; so $\tilde{\mathcal{F}}_0 = \{\emptyset, \tilde{\Omega}\}$. Also, we let $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_t$.

Consider the random variable $Z = f(X_t) : \tilde{\Omega} \rightarrow \mathbb{R}$; note that $f(X_t)$ is $\tilde{\mathcal{F}}_t$ -measurable. Also, for $j = 0, \dots, t$ let Z_j be given by

$$Z_j = \tilde{\mathbb{E}}_{x_0}[f(X_t) | \tilde{\mathcal{F}}_j] = \tilde{\mathbb{E}}_{x_0}[f(X_t) | X_0, \dots, X_j] = (P^{t-j}f)(X_j),$$

where we have used the Markov property in the last equality.

Fix $1 \leq j \leq t$; we want to upper bound $\text{ran}_j = \text{ran}(Z_j | \tilde{\mathcal{F}}_{j-1})$. The σ -field $\tilde{\mathcal{F}}_{j-1}$ can be decomposed into events $\{\tilde{\omega} : \tilde{\omega}_i = x_i \text{ for all } i \leq j-1\}$, for different possible paths x_0, x_1, \dots, x_{j-1} of X . Fix $x_1, \dots, x_{j-1} \in S$, and for $x \in S$ such that $P(x_{j-1}, x) > 0$ consider

$$h(x) = \tilde{\mathbb{E}}_{x_0}[f(X_t) | X_0 = x_0, \dots, X_j = x] = (P^{t-j}f)(x).$$

Note that $Z_j(\tilde{\omega}) \in \{h(x) : P(x_{j-1}, x) > 0\}$ for $\tilde{\omega}$ such that $X_{j-1}(\tilde{\omega}) = x_{j-1}$. It follows from (2.1) that, for such $\tilde{\omega}$,

$$\begin{aligned} \text{ran}_j(\tilde{\omega}) &= \sup_{x, y: P(x_{j-1}, x) > 0, P(x_{j-1}, y) > 0} |h(x) - h(y)| \\ &\leq 2 \sup_{x: P(x_{j-1}, x) > 0} |(P^{t-j}f)(x_{j-1}) - (P^{t-j}f)(x)| \leq 2\alpha_{t-j}. \end{aligned}$$

It follows that

$$R_t^2(\tilde{\omega}) \leq 4 \sum_{i=0}^{t-1} \alpha_i^2,$$

uniformly over $\tilde{\omega} \in \tilde{\Omega}$. Part (i) of Theorem 2.1 now follows from Lemma 2.2.

To prove (ii), observe that the bound

$$\text{ran}_j(\tilde{\omega}) = \text{ran}(Z_j | \tilde{\mathcal{F}}_{j-1})(\tilde{\omega}) \leq 2\alpha_{t-j}$$

still holds on the event $A_t = \{\tilde{\omega} : X_j(\tilde{\omega}) \in S_0^0 \text{ for } j = 0, \dots, t-1\}$. \square

The next, more refined, concentration of measure result, is the one that we actually use in our proofs.

thm.concb-general

Theorem 2.3. *Let P be the transition matrix of a discrete-time Markov chain with discrete state space S . Let $f : S \rightarrow \mathbb{R}$ be a function. Suppose the set S_0 and numbers $\alpha_i(x, y)$ ($x, y \in S_0$) satisfy the following conditions: for all $i \in \mathbb{Z}^+$ and all $x, y \in S_0$,*

$$|\mathbb{E}_x[f(X_i)] - \mathbb{E}_y[f(X_i)]| \leq \alpha_i(x, y). \quad (2.7)$$

cond-gen-3

Let $S_0^0 = \{x \in S_0 : y \in S_0 \text{ whenever } P(x, y) > 0\}$. For $x \in S$ and $i \in \mathbb{Z}^+$, let $a_{x,i} : S \rightarrow \mathbb{R}$ be given by

$$a_{x,i}(y) = \alpha_i(x, y).$$

Assume that, for some sequence $(\alpha_i)_{i \in \mathbb{Z}^+}$ of positive constants,

$$\sup_{x \in S_0^0} (Pa_{x,i}^2)(x) \leq \alpha_i^2. \quad (2.8)$$

cond-gen-4

Let $t > 0$, and let $\beta = 2 \sum_{i=0}^{t-1} \alpha_i^2$. Suppose also that α is such that

$$2 \sup_{0 \leq i \leq t-1} \sup_{x \in S_0^0, y \in N(x)} \alpha_i(x, y) \leq \alpha. \quad (2.9) \quad \boxed{\text{cond-gen-5}}$$

Finally, let $A_t = \{\omega : X_s(\omega) \in S_0^0 : 0 \leq s \leq t-1\}$. Then, for all $a > 0$,

$$\mathbb{P}_{x_0} \left(\{|f(X_t) - \mathbb{E}_{x_0}[f(X_t)]| \geq a\} \cap A_t \right) \leq 2e^{-a^2/(2\beta(1+(\alpha\alpha/3\beta)))}. \quad (2.10) \quad \boxed{\text{ineq.concc}}$$

To prove Theorem 2.3, we use another result from [10].

As before, let Z be a bounded random variable on a finite probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Fix $t \in \mathbb{N}$, and let $\{\emptyset, \tilde{\Omega}\} = \tilde{\mathcal{F}}_0 \subseteq \tilde{\mathcal{F}}_1 \subseteq \dots \subseteq \tilde{\mathcal{F}}_t \subseteq \tilde{\mathcal{F}}$ be a filtration. For $i = 0, \dots, t$, let $Z_i = \tilde{\mathbb{E}}(Z \mid \tilde{\mathcal{F}}_i)$. For $i = 1, \dots, t$, define the *conditional variance*

$$\text{var}_i = \tilde{\text{var}}(Z_i \mid \tilde{\mathcal{F}}_{i-1}) = \tilde{\mathbb{E}}\left((Z_i - \tilde{\mathbb{E}}(Z_i \mid \tilde{\mathcal{F}}_{i-1}))^2 \mid \tilde{\mathcal{F}}_{i-1}\right).$$

Further, let $V = \sum_{i=1}^t \text{var}_i$, the *sum of conditional variances*. Also, for each such $i = 1, \dots, t$, define the *i-th conditional deviation*

$$\text{dev}_i = \sup(|Z_i - Z_{i-1}| \mid \tilde{\mathcal{F}}_{i-1}),$$

and let the *conditional deviation* be $\text{dev} = \max_i \text{dev}_i$. Note that V and dev are random variables in $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The following result is a ‘two-sided’ version (and a simple consequence) of Theorem 3.15 in [10].

thm.mart-b

Lemma 2.4. *Let Z be a random variable on a finite probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\mathbb{E}(Z) = m$. Let $\{\emptyset, \tilde{\Omega}\} = \tilde{\mathcal{F}}_0 \subseteq \tilde{\mathcal{F}}_1 \subseteq \dots \subseteq \tilde{\mathcal{F}}_t$ be a filtration in $\tilde{\mathcal{F}}$, and assume that Z is $\tilde{\mathcal{F}}_t$ -measurable. Let $\hat{\alpha} = \max_{\tilde{\omega} \in \tilde{\Omega}} \text{dev}(\tilde{\omega})$, the maximum conditional deviation. Let $\hat{\beta} = \max_{\tilde{\omega} \in \tilde{\Omega}} V(\tilde{\omega})$, the maximum sum of conditional variances. Assume that $\hat{\alpha}$ and $\hat{\beta}$ are finite. Then for any $a \geq 0$,*

$$\mathbb{P}(|Z - m| \geq a) \leq 2e^{-a^2/(2\hat{\beta}(1+(\hat{\alpha}\hat{\alpha}/3\hat{\beta})))}.$$

More generally, for any $a \geq 0$ and any values $\alpha, \beta \geq 0$,

$$\mathbb{P}(\{|Z - m| \geq a\} \cap \{V \leq \beta\} \cap \{\text{dev} \leq \alpha\}) \leq 2e^{-a^2/(2\beta(1+(\alpha\alpha/3\beta)))}.$$

Proof of Theorem 2.3. We start, as in the proof of the previous theorem, by assuming that $S_0 = S$. Let $f : S \rightarrow \mathbb{R}$ be a function. Fix a time $t \in \mathbb{N}$, and an $x_0 \in S$; consider the evolution of $X = (X_t)_{t \geq 0}$ conditional on $X_0 = x_0$ for t steps, that is until time t . Again this process can be supported by a finite probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_{x_0})$.

For each $j = 0, \dots, t$ let $\tilde{\mathcal{F}}_j = \sigma(X_0, \dots, X_j)$, the σ -field generated by X_0, \dots, X_j ; so $\tilde{\mathcal{F}}_0 = \{\emptyset, \tilde{\Omega}\}$. Also, let $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_t$. We consider the random variable $Z = f(X_t) : \tilde{\Omega} \rightarrow \mathbb{R}$. And, for $j = 0, \dots, t$, Z_j is given by

$$Z_j = \tilde{\mathbb{E}}_{x_0}[f(X_t) \mid \tilde{\mathcal{F}}_j] = \tilde{\mathbb{E}}_{x_0}[f(X_t) \mid X_0, \dots, X_j] = (P^{t-j}f)(X_j).$$

We want to apply Lemma 2.4, so we need to calculate the conditional variances var_i . We use the fact that the variance of a random variable Y is

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equal to $\frac{1}{2} \mathbb{E}(Y - \tilde{Y})^2$, where \tilde{Y} is another random variable with the same distribution as Y and independent of Y .

Fix $1 \leq j \leq t$ and $x_1, \dots, x_{j-1} \in S$, and for $x \in S$ consider

$$h(x) = \tilde{\mathbb{E}}_{x_0} [f(X_t) | X_0 = x_0, \dots, X_{j-1} = x_{j-1}, X_j = x] = (P^{t-j} f)(x).$$

Then, for $\tilde{\omega}$ such that $X_{j-1}(\tilde{\omega}) = x_{j-1}$, $Z_j(\tilde{\omega}) \in \{h(x) : P(x_{j-1}, x) > 0\}$, so that

$$\begin{aligned} \text{var}_j(\tilde{\omega}) &= \frac{1}{2} \sum_{x,y} P(x_{j-1}, x) P(x_{j-1}, y) (h(x) - h(y))^2 \\ &= \frac{1}{2} \sum_{x,y} P(x_{j-1}, x) P(x_{j-1}, y) \left((P^{t-j} f)(x) - (P^{t-j} f)(y) \right)^2 \\ &\leq \sum_{x,y} P(x_{j-1}, x) P(x_{j-1}, y) \left((P^{t-j} f)(x) - (P^{t-j} f)(x_{j-1}) \right)^2 \\ &\quad + \sum_{x,y} P(x_{j-1}, x) P(x_{j-1}, y) \left((P^{t-j} f)(x_{j-1}) - (P^{t-j} f)(y) \right)^2 \\ &\leq 2 \sum_{x: P(x_{j-1}, x) > 0} P(x_{j-1}, x) \left((P^{t-j} f)(x) - (P^{t-j} f)(x_{j-1}) \right)^2 \\ &\leq 2 \sum_x P(x_{j-1}, x) a_{x_{j-1}, t-j}(x)^2 \\ &= 2(Pa_{x_{j-1}, t-j}^2)(x_{j-1}) \leq 2\alpha_{t-j}^2, \end{aligned}$$

by assumption (2.8). It follows that

$$\hat{\beta} \leq \beta = 2 \sum_{i=0}^{t-1} \alpha_i^2.$$

We now bound $\text{dev} = \max_{1 \leq j \leq t} \text{dev}_j$. For $\tilde{\omega}$ such that $X_{j-1}(\tilde{\omega}) = x_{j-1}$,

$$\begin{aligned} \text{dev}_j(\tilde{\omega}) &= \sup_{x: P(x_{j-1}, x) > 0} |(P^{t-j} f)(x) - (P^{t-j+1} f)(x_{j-1})| \\ &= \sup_{x: P(x_{j-1}, x) > 0} |(P^{t-j} f)(x) - (P(P^{t-j} f))(x_{j-1})| \\ &\leq \sup_{x: P(x_{j-1}, x) > 0} \sum_y P(x_{j-1}, y) |(P^{t-j} f)(x) - (P^{t-j} f)(y)| \\ &\leq \sup_{x: P(x_{j-1}, x) > 0} \sum_y P(x_{j-1}, y) |(P^{t-j} f)(x) - (P^{t-j} f)(x_{j-1})| \\ &\quad + \sum_y P(x_{j-1}, y) |(P^{t-j} f)(y) - (P^{t-j} f)(x_{j-1})| \\ &\leq 2 \sup_{x: P(x_{j-1}, x) > 0} |(P^{t-j} f)(x) - (P^{t-j} f)(x_{j-1})| \leq \alpha, \end{aligned}$$

for each $1 \leq j \leq t$, by assumption (2.9). Therefore $\hat{\alpha} \leq \alpha$.

Theorem 2.3 now follows from the first statement in Lemma 2.4 in the case where $S_0 = S$. In general, the above bounds on V and dev hold on the event $A_t = \{\omega : X_i(\omega) \in S_0^0 \text{ for } i = 0, \dots, t-1\}$, and so the full statement of Theorem 2.3 also follows from the second inequality in Lemma 2.4. \square

We now prove that the expectation of a well concentrated function f multiplied by an indicator function approximately factorises, with bounds in terms of bounds on f and its deviations from its mean. This result will be used several times in our proof of Theorems 1.1 and 1.2.

lem. expec

Lemma 2.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a measurable space (S, \mathcal{S}) . Let $f : S \rightarrow \mathbb{R}$ be a measurable function, and suppose that $\mathbb{P}(|f(X) - \mathbb{E}f(X)| \geq a) \leq b$ and $\mathbb{P}(|f(X)| \leq c) = 1$. Let $A \in \mathcal{S}$. Then*

$$\left| \mathbb{E}[\mathbb{I}_{X \in A} f(X)] - \mathbb{P}(X \in A) \mathbb{E}[f(X)] \right| \leq a \mathbb{P}(X \in A) + bc.$$

Proof. We have

$$\begin{aligned} \mathbb{E}[\mathbb{I}_{X \in A} f(X)] &= \mathbb{E}[\mathbb{I}_{X \in A} f(X) \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| \leq a}] \\ &\quad + \mathbb{E}[\mathbb{I}_{X \in A} f(X) \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| > a}]. \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{E}[\mathbb{I}_{X \in A} f(X) \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| \leq a}] &\leq \mathbb{E}[\mathbb{I}_{X \in A} (\mathbb{E}[f(X)] + a)] \\ &= \mathbb{P}(X \in A) \mathbb{E}[f(X)] + \mathbb{P}(X \in A) a, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[\mathbb{I}_{X \in A} f(X) \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| \leq a}] &\geq \mathbb{E}[\mathbb{I}_{X \in A} (\mathbb{E}[f(X)] - a)] \\ &= \mathbb{P}(X \in A) \mathbb{E}[f(X)] - \mathbb{P}(X \in A) a. \end{aligned}$$

Also, $\mathbb{E}[\mathbb{I}_{X \in A} f(X) \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| > a}] \leq c \mathbb{E}[\mathbb{I}_{X \in A} \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| > a}] \leq cb$ and

$$\mathbb{E}[\mathbb{I}_{X \in A} f(X) \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| > a}] \geq -c \mathbb{E}[\mathbb{I}_{X \in A} \mathbb{I}_{|f(X) - \mathbb{E}[f(X)]| > a}] \geq -cb.$$

The result follows. \square

S: gen

3. GENERATOR OF THE MARKOV PROCESS

We now return to the routing model described in the introduction.

Recall that $f_{v,j}(x)$ denotes the number of links with one end v carrying exactly j calls, and that $\mathbb{I}_{uv}^j(x) = 1$ if $x(\{u, v\}) = j$ and $\mathbb{I}_{uv}^j(x) = 0$ otherwise, for all u, v, j . Also, we let $\mathbb{I}_{uv}^{\leq j}(x) = 1$ if $x(\{u, v\}) \leq j$ and $\mathbb{I}_{uv}^{\leq j}(x) = 0$ otherwise. Further, we define $\mathbb{I}_{uv,w}^{\leq j}(x) = 1$ if $x(\{u, w\}) \vee x(\{v, w\}) \leq j$, and $\mathbb{I}_{uv,w}^{\leq j}(x) = 0$ otherwise.

Let A be the generator operator of the Markov process X . By standard theory of Markov chains, for each $t \geq 0$, each $v \in V_n$ and each $j \in \{0, \dots, C\}$,

$$\frac{d \mathbb{E}[f_{v,j}(X_t)]}{dt} = \mathbb{E}[A f_{v,j}(X_t)],$$

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so in order to prove Theorem 1.1 we will need to be able to approximate $\mathbb{E}[Af_{v,j}(X_t)]$.

For $x \in \tilde{S}$ and $0 < j < C$, we can write

$$\begin{aligned} Af_{v,j}(x) &= \lambda f_{v,j-1}(x) - \lambda f_{v,j}(x) + \lambda g_{v,j-1}(x) - \lambda g_{v,j}(x) \\ &\quad - j f_{v,j}(x) + (j+1) f_{v,j+1}(x), \\ Af_{v,0}(x) &= -\lambda f_{v,0}(x) - \lambda g_{v,0}(x) + f_{v,1}(x), \\ Af_{v,C}(x) &= \lambda f_{v,C-1}(x) + \lambda g_{v,C-1}(x) - C f_{v,C}(x), \end{aligned}$$

where the $g_{v,j}(x)$ represent contributions due to alternatively routed arrivals with one end v , and are given, for $j = 0, \dots, C-1$, by

$$\begin{aligned} g_{v,j} &= \frac{1}{(n-2)^d} \left[\sum_{r=1}^d \sum_{u, \mathbf{w}} \mathbb{I}_{uv}^C \mathbb{I}_{vw_r}^j \mathbb{I}_{uw_r}^{\leq j} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv, w_s}^{\leq j}) \prod_{s=r+1}^d (1 - \mathbb{I}_{uv, w_s}^{\leq j-1}) \right. \\ &\quad + \sum_{r=1}^d \sum_{u, \mathbf{w}} \mathbb{I}_{uv}^C \mathbb{I}_{vw_r}^j \sum_{i=j+1}^{C-1} \mathbb{I}_{uw_r}^i \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv, w_s}^{\leq i}) \prod_{s=r+1}^d (1 - \mathbb{I}_{uv, w_s}^{\leq i-1}) \\ &\quad + \sum_{r=1}^d \sum_{u, v', \mathbf{w}_r} \mathbb{I}_{uv'}^C \mathbb{I}_{uv}^j \mathbb{I}_{v'v}^{\leq j} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv', w_s}^{\leq j}) \prod_{s=r+1}^d (1 - \mathbb{I}_{uv', w_s}^{\leq j-1}) \\ &\quad \left. + \sum_{r=1}^d \sum_{u, v', \mathbf{w}_r} \mathbb{I}_{uv'}^C \mathbb{I}_{uv}^j \sum_{i=j+1}^{C-1} \mathbb{I}_{v'v}^i \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv', w_s}^{\leq i}) \prod_{s=r+1}^d (1 - \mathbb{I}_{uv', w_s}^{\leq i-1}) \right], \quad (3.1) \end{aligned}$$

with $\mathbf{w} = (w_1, \dots, w_d) \in V_n^d$, $\mathbf{w}_r = (w_1, \dots, w_{r-1}, w_{r+1}, \dots, w_d) \in V_n^{d-1}$. Here, $\sum_{u, \mathbf{w}}$ denotes the sum over all $u \neq v$, and over all w_1, \dots, w_d such that each $w_r \neq u, v$, and $\sum_{u, v', \mathbf{w}_r}$ denotes the sum over all $u \neq v$, $v' \neq u, v$ and over all $w_1, \dots, w_{r-1}, w_{r+1}, \dots, w_d$ such that each $w_j \neq u, v'$.

In (3.1), the first term is the probability that the direct link chosen for a new call with one end v is blocked and, on the two-link path selected for the call, the link including v has load j , while its partner link has load at most j . The second term is the probability that the direct link chosen for a new call with one end v is blocked and, on the two-link path selected for the call, the link including v has load j and its partner link has load greater than j . The third term is the probability that v is chosen as an intermediate node for a call blocked from its direct link, the route through v is the best out of the d routes selected and j is the maximum load of a link on the route. The fourth term is the probability that v is chosen as an intermediate node for a call blocked from its direct link, the route through v is the best of the d routes selected, and j is not the maximum load of a link on the route.

In particular, when $d = 1$, for $j = 0, \dots, C-1$, for each $v \in V_n$,

$$g_{v,j} = \frac{1}{n-2} \sum_{u, w \in V_n} (\mathbb{I}_{uv}^C \mathbb{I}_{vw}^j \mathbb{I}_{uw}^{\leq C-1} + \mathbb{I}_{uw}^C \mathbb{I}_{vw}^j \mathbb{I}_{uv}^{\leq C-1}).$$

Furthermore, when $d = 2$, then $g_{v,j}$ is a sum of 8 contributions. These contributions correspond to the case where v is an end node and the case where v is an intermediate node for a call. In the case where v is an end node, we have a subcase where the first route of those selected is allocated to a new call and a subcase where the second route of those selected is allocated to a new call. We further need to distinguish a subcase where the link ending in v has the maximum load, and a subcase where the link ending in v does not have the maximum load on the route allocated to a new call. In the case where v is an intermediate node, we need to distinguish a subcase where v is the first alternative node selected and a subcase where v is the second alternative node selected. Also, we have a subcase where the link with load j has the maximum load, and a subcase where a link with load j does not have the maximum load on the route allocated to a new call.

The contribution due to the case where v is an end node, the first route of those selected is allocated to a new call, and the link ending in v has the maximum load is of the form

$$\frac{1}{(n-2)^2} \sum_{u, w_1, w_2 \in V_n} \mathbb{I}_{uv}^C \mathbb{I}_{vw_1}^j \mathbb{I}_{uw_1}^{\leq j} (1 - \mathbb{I}_{uv, w_2}^{\leq j-1}).$$

The contribution due to the case where v is an end node, the second route of those selected is allocated to a new call, and the link ending in v does not have the maximum load is of the form

$$\frac{1}{(n-2)^2} \sum_{u, w_1, w_2 \in V_n} \mathbb{I}_{uv}^C \mathbb{I}_{vw_2}^j \sum_{i=j+1}^{C-1} \mathbb{I}_{uw_2}^i (1 - \mathbb{I}_{uv, w_1}^{\leq i}).$$

The contribution due to the case where v is an intermediate node and is selected first, and where a link with load j has the maximum load on the route allocated to a new call is of the form

$$\frac{1}{(n-2)^2} \sum_{u, v', w \in V_n} \mathbb{I}_{uv'}^C \mathbb{I}_{uv}^j \mathbb{I}_{v'v}^{\leq j} (1 - \mathbb{I}_{uv', w}^{\leq j-1}).$$

The remaining contributions can be expressed analogously, and the form of g for $d > 2$ is derived similarly.

We note that each $g_{v,j}$ is a sum of products of indicators of sets of load vectors with properties pertaining to loads of particular links. Our plan is to justify the intuition that, subject to suitable initial conditions, the loads on different links behave nearly as iid random variables at each time t , and the precise estimates we use involve sums over reasonably large collections of links. We need several specific manifestations of this near-independence and symmetry. First, the geometry of the network is not important; this means that, for fixed nodes u and v , the loads on links uw and vw are not strongly correlated, on average over w , so that the average value of $\mathbb{I}_{uw}^j \mathbb{I}_{vw}^k$ over w is close to the product of the average values of \mathbb{I}_{uw}^j and \mathbb{I}_{vw}^k . (In other words, the function ϕ^1 defined earlier is small.) Secondly, for fixed nodes u and v , the loads on links incident on u have approximately the

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same distribution as the loads on links incident on v . (This means that the function ϕ^2 is small.) Thirdly, we require that the alternatively routed calls are fairly uniformly distributed over the network. (This implies that the function ϕ^3 is small.) Finally, we will show that each $f_{v,j}(X_t)$ is well concentrated around its expectation, which then implies that, for fixed nodes u and v , $\mathbb{E}[f_{u,j}(X_t)f_{v,j}(X_t)]$ is approximately equal to $\mathbb{E}[f_{u,j}(X_t)]\mathbb{E}[f_{v,j}(X_t)]$. Naturally, quantitative versions of these properties will need to be assumed to hold at time 0, and we will show that they are maintained throughout the time period of interest. This will then allow us to express $\mathbb{E}[g_{v,j}(X_t)]$ as a (scaled) sum of products of terms of the form $\frac{1}{n-1}f_{v,i}(t)$, and hence lead to approximate differential equations satisfied by the $\mathbb{E}[f_{v,j}(X_t)]$ for $j = 0, 1, \dots, C$, for each $v \in V_n$, expressed in terms of themselves.

Let $f_{v,\leq j}(x) = \sum_{i \leq j} f_{v,i}(x)$. Let $f_{v,j}(t) = \mathbb{E}[f_{v,j}(X_t)]$, and let $f_{v,\leq j}(t) = \mathbb{E}[f_{v,\leq j}(X_t)]$. Let $\mathbb{I}_{uv}^j(t) = \mathbb{E}[\mathbb{I}_{uv}^j(X_t)]$ and let $\mathbb{I}_{uv}^{\leq j}(t) = \mathbb{E}[\mathbb{I}_{uv}^{\leq j}(X_t)]$. We will show that the expectation of the first term in (3.1) with respect to the law of X_t is approximately

$$\begin{aligned}
& \frac{1}{(n-2)^d} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^C(t) \mathbb{E} \left[\left(\sum_{\mathbf{w}} \mathbb{I}_{vw_r}^j \mathbb{I}_{uw_r}^{\leq j} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv,w_s}^{\leq j}) \right. \right. \\
& \quad \left. \left. \times \prod_{s=r+1}^d (1 - \mathbb{I}_{uv,w_s}^{\leq j-1}) \right) (X_t) \right] \\
& \approx \frac{1}{(n-1)^{2d}} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^C(t) \mathbb{E} \left[\left(\sum_{\mathbf{w}, \mathbf{w}'} \mathbb{I}_{vw_r}^j \mathbb{I}_{uw_r'}^{\leq j} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw_s'}^{\leq j}) \right. \right. \\
& \quad \left. \left. \times \prod_{s=r+1}^d (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw_s'}^{\leq j-1}) \right) (X_t) \right] \\
& \approx \frac{1}{(n-1)^{2d}} f_{v,C}(t) f_{v,j}(t) f_{v,\leq j}(t) \sum_{r=1}^d (1 - (f_{v,\leq j}(t))^2)^{r-1} \\
& \quad \times (1 - (f_{v,\leq j-1}(t))^2)^{d-r}.
\end{aligned}$$

Handling the remaining terms in a similar way, we will prove that

$$\begin{aligned}
\mathbb{E}[g_{v,j}(X_t)] & \approx \frac{2}{(n-1)^{2d}} f_{v,C}(t) f_{v,j}(t) f_{v,\leq j}(t) \sum_{r=1}^d (1 - (f_{v,\leq j}(t))^2)^{r-1} \\
& \quad \times (1 - (f_{v,\leq j-1}(t))^2)^{d-r} \\
& \quad + \frac{2}{(n-1)^{2d}} f_{v,C}(t) f_{v,j}(t) \sum_{i=j+1}^{C-1} f_{v,i}(t) \sum_{r=1}^d (1 - (f_{v,\leq i}(t))^2)^{r-1} \\
& \quad \times (1 - (f_{v,\leq i-1}(t))^2)^{d-r}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{(n-1)^{2d}} f_{v,C}(t) f_{v,j}(t) f_{v,\leq j}(t) \\
&\quad \times \frac{(1 - (f_{v,\leq j-1}(t))^2)^d - (1 - (f_{v,\leq j}(t))^2)^d}{f_{v,\leq j}(t)^2 - (f_{v,\leq j-1}(t))^2} \\
&\quad + \frac{2}{(n-1)^{2d}} f_{v,C}(t) f_{v,j}(t) \sum_{i=j+1}^{C-1} f_{v,i}(t) \\
&\quad \times \frac{(1 - (f_{v,\leq i-1}(t))^2)^d - (1 - (f_{v,\leq i}(t))^2)^d}{f_{v,\leq i}(t)^2 - (f_{v,\leq i-1}(t))^2}.
\end{aligned}$$

Hence we will see that the functions $\mathbb{E}[f_{v,j}(X_t)]$ ($j = 0, \dots, C$, $v \in V_n$) approximately solve the differential equation (1.3). As the $f_{v,j}(X_t)$ are well concentrated around their expectations, Theorems 1.1 and 1.2 will follow.

sec:couple

4. COUPLING

In this section, we describe and analyse a natural coupling between two copies of process X . We start by defining notions of ‘distance’ between the two copies, with the aim of showing that the expected distance grows slowly in time, at least for time $O(1)$.

Given two load vectors x, y , the ℓ_1 -distance between them is

$$\|x - y\|_1 = \sum_{e \in E_n} |x(e, 0) - y(e, 0)| + \sum_{e \in E_n, w \notin e} |x(e, w) - y(e, w)|,$$

which measures the sum of the differences between x and y in loads of all the different routes. Then $\|\cdot\|$ is a metric on S .

For $v \in V_n$ we will also consider functions

$$\begin{aligned}
\|x - y\|_v &= \sum_{e: v \in e} |x(e, 0) - y(e, 0)| + \sum_{e: v \in e} \sum_{w \notin e} |x(e, w) - y(e, w)| \\
&\quad + \sum_{e: v \notin e} |x(e, v) - y(e, v)| \\
&= \sum_{u \neq v} |x(\{u, v\}, 0) - y(\{u, v\}, 0)| \\
&\quad + \sum_{u \neq v} \sum_{w \neq u, v} |x(\{u, v\}, w) - y(\{u, v\}, w)| \\
&\quad + \sum_{\{u, w\}: u \neq w, v \notin \{u, w\}} |x(\{u, w\}, v) - y(\{u, w\}, v)|.
\end{aligned}$$

Then $2\|x - y\|_v$ gives an upper bound on the sum of the differences between the loads of links $\{v, w\}$, $w \neq v$ (i.e. links around node v) in x and y .

For the remainder of this section, and subsequently, we will work with a jump chain \hat{X} corresponding to X . The chain we use is not the standard embedded chain but a slower moving version that will often not change its state at a given step. Since we only work on the set \tilde{S} , we can use a jump

chain with transitions defined as follows. Given that the current state, at time $t \in \mathbb{Z}^+$, is $x \in \tilde{S}$, the next event is an arrival with probability

$$p(\lambda, n) = \frac{\lambda \binom{n}{2}}{\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor}$$

and a *potential* departure with probability $1 - p(\lambda, n)$. Given that the event is an arrival, each pair of endpoints u, v is chosen with probability $1/\binom{n}{2}$, then each d -tuple of intermediate nodes is chosen with probability $(n-2)^{-d}$, and the call is routed along the two-link route chosen first among the d selected that minimises the maximum load of a link. Given that the event is a potential departure, the calls currently in the system are enumerated from 1 up to at most $\lfloor 6\lambda \binom{n}{2} \rfloor$, and then a number is chosen uniformly at random from $\{1, \dots, \lfloor 6\lambda \binom{n}{2} \rfloor\}$. If there is a call assigned to this number, it departs, and otherwise nothing happens.

Let $S_0 \subseteq \tilde{S}$ be the set of states x such that $\|x\|_1 \leq 4\lambda \binom{n}{2}$. Recall also that $S_1 \subseteq S_0$ is the set of states x such that $\|x\|_1 \leq 2\lambda \binom{n}{2}$. We will be interested in the evolution of the chain starting from S_1 and before it leaves S_0 .

Consider the following family of Markovian couplings $(\hat{X}^{x_0}, \hat{Y}^{y_0})$ of pairs of copies $\hat{X}^{x_0}, \hat{Y}^{y_0}$ of the discrete jump chain starting from states x_0, y_0 respectively, where $x_0, y_0 \in S_0$. (In what follows, we will drop the superscripts x_0, y_0 from the notation and refer simply to \hat{X} and \hat{Y} .)

Let $t \geq 0$, and let x, y be both in \tilde{S} . Given that $\hat{X}_{t-1} = x$ and $\hat{Y}_{t-1} = y$, the transition at time t (from state $(\hat{X}_{t-1}, \hat{Y}_{t-1})$ to (\hat{X}_t, \hat{Y}_t)) is an arrival in both \hat{X} and \hat{Y} , or a potential departure in both \hat{X} and \hat{Y} . Given that the transition is an arrival, we choose the same call endpoints and the same d -tuple of intermediate nodes in both. Also, given that the transition is a potential departure, we pair the calls occupying the same route in both \hat{X} and \hat{Y} , as much as possible. We also pair off the remaining calls arbitrarily, as much as possible. (We can pair off all the calls if $\|x\|_1 = \|y\|_1$, but otherwise some calls will remain unpaired in the process that currently has more calls.) Then we always choose the paired calls for simultaneous departure; any calls that are unpaired will depart alone. This can be achieved by assigning to each pair, and to each unpaired call, a distinct number in $\{1, \dots, \lfloor 6\lambda \binom{n}{2} \rfloor\}$. If the transition at time t is a potential departure, we choose the same uniformly random number from $\{1, \dots, \lfloor 6\lambda \binom{n}{2} \rfloor\}$ for both \hat{X} and \hat{Y} . If the number corresponds to a pair of calls, both depart; if it corresponds to an unpaired call, this call departs; otherwise, nothing happens.

Suppose now $\hat{X}_{t-1} = x$ and $\hat{Y}_{t-1} = y$ where at least one of x and y , say x , is not in \tilde{S} . We then let $\hat{X}_t = \hat{X}_{t-1} = x$, while \hat{Y}_t is obtained from \hat{Y}_{t-1} by following the transition probabilities of the jump chain from y if $y \in \tilde{S}$ or $\hat{Y}_t = \hat{Y}_{t-1} = y$ if $y \notin \tilde{S}$.

The process $\hat{W} = (\hat{W}_t)$ given by $\hat{W}_t = (\hat{X}_t, \hat{Y}_t)$ is a Markov chain adapted to its natural filtration, $\mathcal{G}_t = \sigma(\hat{X}_s, \hat{Y}_s : s \leq t)$.

Note that, on the event that the jump at time t is a potential departure, $\|\hat{X}_t - \hat{Y}_t\|_1 \leq \|x - y\|_1$. (The distance remains unchanged if paired calls from the same route depart or if there is no departure at all; it decreases by 2 if paired calls on different routes depart, and decreases by 1 if an unpaired call departs.) Furthermore, the distance between \hat{X} and \hat{Y} can only increase by 2 at an event, and this can only happen if the jump is an arrival and if we select at least one of the links where \hat{X}_{t-1} and \hat{Y}_{t-1} differ. This happens with probability at most

$$\frac{2d+1}{\binom{n}{2}} \sum_{e \in E_n} |\hat{X}_{t-1}(e) - \hat{Y}_{t-1}(e)| \leq \frac{3d}{\binom{n}{2}} \sum_{e \in E_n} |\hat{X}_{t-1}(e) - \hat{Y}_{t-1}(e)|,$$

and $\sum_{e \in E_n} |\hat{X}_{t-1}(e) - \hat{Y}_{t-1}(e)|$ is equal to

$$\begin{aligned} & \sum_{\{u,v\}: u \neq v} |(\hat{X}_{t-1}(\{u,v\}, 0) - \hat{Y}_{t-1}(\{u,v\}, 0)) \\ & + \sum_{w \neq u,v} (\hat{X}_{t-1}(\{v,w\}, u) - \hat{Y}_{t-1}(\{v,w\}, u)) \\ & + \sum_{w \neq u,v} (\hat{X}_{t-1}(\{u,w\}, v) - \hat{Y}_{t-1}(\{u,w\}, v))| \leq 2\|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_1. \end{aligned}$$

It follows that, uniformly over all $x, y \in \tilde{S}$,

$$\mathbb{E} \left[\|\hat{X}_t - \hat{Y}_t\|_1 \mid \hat{X}_{t-1} = x, \hat{Y}_{t-1} = y \right] \leq \left(1 + \frac{12d}{\binom{n}{2}} \right) \|x - y\|_1.$$

We have assumed that $\hat{X}_0 = x_0$ and $\hat{Y}_0 = y_0$, where $x_0, y_0 \in S_0$, that is $\|\hat{X}_0\|_1 \leq 4\lambda\binom{n}{2}$ and $\|\hat{Y}_0\|_1 \leq 4\lambda\binom{n}{2}$. Note that, whenever $\|\hat{X}_{t-1}\|_1 \geq 4\lambda\binom{n}{2}$, provided $n \geq \max(3, \frac{1}{\lambda})$,

$$\mathbb{P}[\|\hat{X}_t\|_1 - \|\hat{X}_{t-1}\|_1 = 1 \mid \hat{X}_{t-1}] \leq \frac{\lambda\binom{n}{2}}{\lambda\binom{n}{2} + \lfloor 6\lambda\binom{n}{2} \rfloor} \leq \frac{1}{6},$$

and

$$\mathbb{P}[\|\hat{X}_t\|_1 - \|\hat{X}_{t-1}\|_1 = -1 \mid \hat{X}_{t-1}] \geq \frac{4\lambda\binom{n}{2}}{\lambda\binom{n}{2} + \lfloor 6\lambda\binom{n}{2} \rfloor} \geq \frac{4}{7} \geq \frac{1}{2}.$$

Therefore, by standard inequalities (see, for instance, Lemma 2.5 in [7], with $p = 1/6$, $q = 1/2$ and $a = 2\lambda\binom{n}{2}$), for any constant $c > 0$,

$$\mathbb{P} \left(\exists t \leq cn^2 : \|\hat{X}_t\|_1 \vee \|\hat{Y}_t\|_1 \geq 6\lambda\binom{n}{2} \right) \leq 2cn^2 \left(\frac{1}{3} \right)^{2\lambda\binom{n}{2}}.$$

Let A_s be the event $\{\hat{X}_u \in \tilde{S}, \hat{Y}_u \in \tilde{S} \text{ for all } u \leq s\}$. Then, for $x_0, y_0 \in S_0$,

$$\begin{aligned} \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 \mathbb{I}_{A_{t-1}}) &= \mathbb{E} \left[\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 \mathbb{I}_{A_{t-1}} \mid \mathcal{G}_{t-1}) \right] \\ &\leq \mathbb{E} \left[\left(1 + \frac{12d}{\binom{n}{2}} \right) \|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_1 \mathbb{I}_{A_{t-1}} \right] \end{aligned}$$

$$\leq \left(1 + \frac{12d}{\binom{n}{2}}\right) \mathbb{E}(\|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_1 \mathbb{I}_{A_{t-2}}).$$

By induction,

$$\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 \mathbb{I}_{A_{t-1}}) \leq \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_1.$$

Since the chain stops once it leaves \tilde{S} and each jump changes $\|\hat{X}_t\|_1$ and $\|\hat{Y}_t\|_1$ by at most 1, on the event $\overline{A_{t-1}}$, $\|\hat{X}_t - \hat{Y}_t\|_1 \leq 2(6\lambda \binom{n}{2} + 1)$. Hence, for any constant $c \leq n$, and any $t \leq cn^2$,

$$\begin{aligned} \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1) &= \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 \mathbb{I}_{A_{t-1}}) + \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 (1 - \mathbb{I}_{A_{t-1}})) \\ &\leq \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_1 + 2(6\lambda \binom{n}{2} + 1) 2cn^2 \left(\frac{1}{3}\right)^{2\lambda \binom{n}{2}} \\ &\leq \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_1 + 14\lambda n^5 \left(\frac{1}{3}\right)^{\frac{1}{2}\lambda n^2}. \end{aligned}$$

If $n \geq \max(1000, 1/\lambda)$, this last term is at most 1. Therefore, for $n \geq \max(1000, 1/\lambda, c)$, and $t \leq cn^2$,

$$\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1) \leq 2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_1, \quad (4.1) \quad \boxed{\text{eq.dist}}$$

uniformly over all possible starting states x_0, y_0 in S_0 .

Let v be a vertex. While in \tilde{S} , we can only change the loads of links at v (i.e. links $\{v, w\}$, for $w \neq v$) if we choose a link with end v at a jump time. Also, we can only make $\|\hat{X}_t - \hat{Y}_t\|_v$ bigger than $\|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_v$ at an arrival time, if either we pick one of the links $\{v, w\}$ (if any) in which \hat{X}_{t-1} and \hat{Y}_{t-1} differ, or if we pick a link $\{u, w\}$ (where $u \neq v$ and $w \neq u, v$) in which \hat{X}_{t-1} and \hat{Y}_{t-1} differ, and also vertex v as an endpoint or an intermediate node for a new call. The former happens with conditional probability at most

$$\begin{aligned} &\frac{2d+1}{\binom{n}{2}} \sum_{w \neq v} \left| \hat{X}_{t-1}(\{v, w\}) - \hat{Y}_{t-1}(\{v, w\}) \right| \\ &\leq \frac{3d}{\binom{n}{2}} \sum_{w \neq v} \left| \hat{X}_{t-1}(\{v, w\}) - \hat{Y}_{t-1}(\{v, w\}) \right|, \end{aligned}$$

and

$$\begin{aligned} &\sum_{w \neq v} \left| \hat{X}_{t-1}(\{v, w\}) - \hat{Y}_{t-1}(\{v, w\}) \right| \\ &= \sum_{w \neq v} \left| \left(\hat{X}_{t-1}(\{v, w\}, 0) - \hat{Y}_{t-1}(\{v, w\}, 0) \right) \right. \\ &\quad \left. + \sum_{u \neq v, w} (\hat{X}_{t-1}(\{u, v\}, w) - \hat{Y}_{t-1}(\{u, v\}, w)) \right| \end{aligned}$$

$$+ \sum_{u \neq v, w} (\hat{X}_{t-1}(\{u, w\}, v) - \hat{Y}_{t-1}(\{u, w\}, v)) \Big| \leq 2 \|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_v.$$

The latter happens with conditional probability at most

$$\begin{aligned} \frac{1}{n} \frac{2d^2}{\binom{n}{2} - (n-1)} \sum_{\{u, w\} \in E_n: v \notin \{u, w\}} |\hat{X}_{t-1}(\{u, w\}) - \hat{Y}_{t-1}(\{u, w\})| \\ \leq \frac{8d^2}{n \binom{n}{2}} \|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_1, \end{aligned}$$

provided $n \geq 4$. Also, always with probability 1,

$$\|\hat{X}_t - \hat{Y}_t\|_v \leq \|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_v + 2.$$

Then, for $n \geq 4$,

$$\begin{aligned} \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_v \mathbb{I}_{A_{t-1}}) &\leq \left(1 + \frac{12d}{\binom{n}{2}}\right) \mathbb{E}(\|\hat{X}_{t-1} - \hat{Y}_{t-1}\|_v \mathbb{I}_{A_{t-2}}) \\ &\quad + \frac{16d^2}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{t-1} \|x_0 - y_0\|_1, \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_v \mathbb{I}_{A_{t-1}}) &\leq \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_v \\ &\quad + \frac{16d^2 t}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{t-1} \|x_0 - y_0\|_1. \end{aligned}$$

Hence, as before, for all $n \geq \max(1000, 1/\lambda, c)$ and $t \leq cn^2$, for each $v \in V_n$, and for all $x_0, y_0 \in S_0$,

$$\begin{aligned} \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_v) &\leq 2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_v \\ &\quad + \frac{32d^2 t}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{t-1} \|x_0 - y_0\|_1. \end{aligned} \quad (4.2)$$

Recall that, for a load vector x , $f_{v,k}(x)$ is the number of links around v carrying exactly k calls. Similarly, $f_{v,\leq k}(x) = \sum_{i \leq k} f_{v,i}(x)$ is the number of links $\{v, w\}$, $w \neq v$, such that $x(\{v, w\}) \leq k$; that is, the number of links around v carrying at most k calls. Let P denote the transition matrix of the jump chain (\hat{X}_t) restricted to \tilde{S} .

Note that, for each v, k , and each $x, y \in \tilde{S}$,

$$\begin{aligned} |f_{v,k}(x) - f_{v,k}(y)| &\leq \sum_{w \neq v} |x(\{v, w\}) - y(\{v, w\})| \leq 2\|x - y\|_v, \\ |f_{v,\leq k}(x) - f_{v,\leq k}(y)| &\leq \sum_{w \neq v} |x(\{v, w\}) - y(\{v, w\})| \leq 2\|x - y\|_v. \end{aligned}$$

A QUANTITATIVE DIFFERENTIAL EQUATION APPROXIMATION

Hence, for $x_0, y_0 \in S_0$, for $n \geq \max(1000, 1/\lambda, c)$, for $t \leq cn^2$ and each v, k ,

$$\begin{aligned} |(P^t f_{v,k})(x_0) - (P^t f_{v,k})(y_0)| &\leq \mathbb{E} |f_{v,k}(\hat{X}_t) - f_{v,k}(\hat{Y}_t)| \leq 2 \mathbb{E} \|\hat{X}_t - \hat{Y}_t\|_v \\ &\leq 4 \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_v \\ &\quad + \frac{64d^2 t}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{t-1} \|x_0 - y_0\|_1. \end{aligned} \quad (4.3)$$

Similarly, for $x_0, y_0 \in S_0$, $n \geq \max(1000, 1/\lambda, c)$, $t \leq cn^2$ and each v, k ,

$$\begin{aligned} |(P^t f_{v,\leq k})(x_0) - (P^t f_{v,\leq k})(y_0)| &\leq 4 \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_v \\ &\quad + \frac{64d^2 t}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{t-1} \|x_0 - y_0\|_1. \end{aligned} \quad (4.4)$$

Given $u, v \in V_n$ and $j, k \in \{0, 1, \dots, C\}$, set

$$\begin{aligned} f_{u,v,j,k} &= \frac{1}{(n-2)^{2d-1}} \sum_{r=1}^d \sum_{w_r} \mathbb{I}_{uw_r}^j \sum_{w'_r} \mathbb{I}_{vw'_r}^k \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \\ &\quad \times \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}); \\ f_{u,v,\leq j,k} &= \frac{1}{(n-2)^{2d-1}} \sum_{r=1}^d \sum_{w_r} \mathbb{I}_{uw_r}^{\leq j} \sum_{w'_r} \mathbb{I}_{vw'_r}^k \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \\ &\quad \times \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}), \end{aligned}$$

where the sums are over all $w_r, w'_r, w_s, w'_s \neq u, v$. Then

$$\begin{aligned} f_{u,v,j,k} &= \frac{1}{(n-2)^{2d-1}} (f_{u,j} - \mathbb{I}_{uv}^j) (f_{v,k} - \mathbb{I}_{uv}^k) \\ &\quad \times \sum_{r=1}^d \left((n-2)^2 - (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j}) (f_{v,\leq j} - \mathbb{I}_{uv}^{\leq j}) \right)^{r-1} \\ &\quad \times \left((n-2)^2 - (f_{u,\leq j-1} - \mathbb{I}_{uv}^{\leq j-1}) (f_{v,\leq j-1} - \mathbb{I}_{uv}^{\leq j-1}) \right)^{d-r}, \\ f_{u,v,\leq j,k} &= \frac{1}{(n-2)^{2d-1}} (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j}) (f_{v,k} - \mathbb{I}_{uv}^k) \\ &\quad \times \sum_{r=1}^d \left((n-2)^2 - (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j}) (f_{v,\leq j} - \mathbb{I}_{uv}^{\leq j}) \right)^{r-1} \\ &\quad \times \left((n-2)^2 - (f_{u,\leq j-1} - \mathbb{I}_{uv}^{\leq j-1}) (f_{v,\leq j-1} - \mathbb{I}_{uv}^{\leq j-1}) \right)^{d-r}. \end{aligned}$$

In the case $d = 1$, we have

$$\begin{aligned} f_{u,v,j,k}(x) &= \frac{1}{n-2} \sum_{w \neq u,v} \mathbb{I}_{uw}^j(x) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(x), \\ f_{u,v,\leq j,k}(x) &= \frac{1}{n-2} \sum_{w \neq u,v} \mathbb{I}_{uw}^{\leq j}(x) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(x), \end{aligned}$$

so that

$$\begin{aligned} f_{u,v,j,k} &= \frac{1}{n-2} (f_{u,j} - \mathbb{I}_{uv}^j)(f_{v,k} - \mathbb{I}_{uv}^k), \\ f_{u,v,\leq j,k} &= \frac{1}{n-2} (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j})(f_{v,k} - \mathbb{I}_{uv}^k). \end{aligned}$$

Then

$$\begin{aligned} &|f_{u,v,j,k}(x) - f_{u,v,j,k}(y)| \\ &\leq \frac{1}{n-2} (f_{v,k}(x) - \mathbb{I}_{uv}^k(x)) |f_{u,j}(x) - \mathbb{I}_{uv}^j(x) - (f_{u,j}(y) - \mathbb{I}_{uv}^j(y))| \\ &\quad + \frac{1}{n-2} (f_{u,j}(y) - \mathbb{I}_{uv}^j(y)) |f_{v,k}(x) - \mathbb{I}_{uv}^k(x) - (f_{v,k}(y) - \mathbb{I}_{uv}^k(y))| \\ &\leq 2\|x - y\|_u + 2\|x - y\|_v. \end{aligned}$$

Similarly,

$$|f_{u,v,\leq j,k}(x) - f_{u,v,\leq j,k}(y)| \leq 2\|x - y\|_u + 2\|x - y\|_v.$$

A calculation similar to the one above shows that, for any $d \geq 1$, if f is one of the functions $f_{u,v,j,k}$, $f_{u,v,\leq j,k}$, then

$$|f(x) - f(y)| \leq 2d^2(\|x - y\|_u + \|x - y\|_v).$$

As before, if f is one of the functions $f_{u,v,j,k}$, $f_{u,v,\leq j,k}$, then, for all $x_0, y_0 \in S_0$, all $n \geq \max(1000, 1/\lambda, c)$, and all $t \leq cn^2$,

$$\begin{aligned} &|(P^t f)(x_0) - (P^t f)(y_0)| \\ &\leq 4d^2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_v + 4d^2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^t \|x_0 - y_0\|_u \\ &\quad + \frac{128d^4 t}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{t-1} \|x_0 - y_0\|_1. \end{aligned} \tag{4.5}$$

5. CONCENTRATION OF MEASURE FOR THE ROUTING MODEL

sec:conc-route

We will now apply Theorem 2.3 to the jump Markov chain \hat{X} and functions $f_{v,k}$, $f_{v,\leq k}$, $f_{u,v,j,k}$, $f_{u,v,\leq j,k}$.

From now on, we assume that our process starts in some fixed state $X_0 = x_0 \in S_1$. We write \mathbb{P} and \mathbb{E} when discussing probabilities relating to \hat{X} , instead of \mathbb{P}_{x_0} and \mathbb{E}_{x_0} , which was convenient in the derivation of the concentration inequalities in Section 2.

A QUANTITATIVE DIFFERENTIAL EQUATION APPROXIMATION

We start with the functions $f_{v,k}$. By (4.3), for all $x, y \in S_0$, we can take

$$\alpha_i(x, y) = 4 \left(1 + \frac{12d}{\binom{n}{2}}\right)^i \|x - y\|_v + \frac{64d^2 i}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{i-1} \|x - y\|_1,$$

for $i \leq cn^2$ and $n \geq \max(1000, 1/\lambda, c)$.

Note that, for all x, y such that $P(x, y) > 0$, we have $\|x - y\|_v \leq 1$ and $\|x - y\|_1 \leq 1$. Also, for each $x \in S_0$,

$$\sum_{y: \|x-y\|_v > 0} P(x, y) \leq \frac{2+d}{n}.$$

It follows that for each $x \in S_0$, for $i \leq cn^2$ and $n \geq \max(1000, 1/\lambda, c)$,

$$\begin{aligned} (Pa_{x,i}^2)(x) &\leq \frac{32(2+d)}{n} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{2i} + 2^{13} \left(\frac{d^2 i}{n \binom{n}{2}}\right)^2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^{2i-2} \\ &\leq \frac{32(2+d)}{n} e^{96dc} + 2^{16} \left(\frac{d^2 c}{n}\right)^2 e^{96dc} \leq \frac{2^{17} d^4 (c+1)^2}{n} e^{96dc}. \end{aligned}$$

So we can take $\alpha_i^2 = \frac{2^{17} d^4 (c+1)^2}{n} e^{96dc}$, and so $\beta \leq 2^{18} d^4 (c+1)^3 n e^{96dc}$, for $t \leq cn^2$. Also, for $n \geq \max(1000, 1/\lambda, 64d^2 c)$ and $t \leq cn^2$, we can take

$$\alpha = 4e^{48dc} + \frac{256d^2 c}{n} e^{48dc} \leq 8e^{48dc}.$$

By Theorem 2.3, for $t \leq cn^2$,

$$\begin{aligned} \mathbb{P} \left(\{|f_{v,k}(\hat{X}_t) - \mathbb{E}[f_{v,k}(\hat{X}_t)]| \geq a\} \cap \{X_s \in S_0^0 : 0 \leq s \leq t-1\} \right) \\ \leq 2e^{-a^2/(2^{19} d^4 (c+1)^3 n e^{96dc} + 32e^{48dc} a/3)}. \end{aligned} \quad (5.1)$$

Similarly,

$$\begin{aligned} \mathbb{P} \left(\{|f_{v,\leq k}(\hat{X}_t) - \mathbb{E}[f_{v,\leq k}(\hat{X}_t)]| \geq a\} \cap \{X_s \in S_0^0 : 0 \leq s \leq t-1\} \right) \\ \leq 2e^{-a^2/(2^{19} d^4 (c+1)^3 n e^{96dc} + 32e^{48dc} a/3)}. \end{aligned} \quad (5.2)$$

We now consider functions $f_{u,v,j,k}$ and $f_{u,v,\leq j,k}$. By (4.5), for all $x, y \in S_0$,

$$\begin{aligned} \alpha_i(x, y) &= 4d^2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^i \|x - y\|_v + 4d^2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^i \|x - y\|_u \\ &\quad + \frac{128d^4 i}{n \binom{n}{2}} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{i-1} \|x - y\|_1, \end{aligned}$$

for $i \leq cn^2$ and $n \geq \max(1000, 1/\lambda, c)$. This leads to

$$\begin{aligned} (Pa_{x,i}^2)(x) &\leq \frac{64d^4 (d+2)}{n} \left(1 + \frac{12d}{\binom{n}{2}}\right)^{2i} + 2^{15} \left(\frac{d^4 i}{n \binom{n}{2}}\right)^2 \left(1 + \frac{12d}{\binom{n}{2}}\right)^{2i-2} \\ &\leq \frac{64d^4 (d+2)}{n} e^{96dc} + 2^{18} \left(\frac{d^4 c}{n}\right)^2 e^{96dc} \leq \frac{2^{19} d^8 (c+1)^2}{n} e^{96dc}, \end{aligned}$$

for each $x \in S_0$, for $t \leq cn^2$ and $n \geq \max(1000, 1/\lambda, 64d^2 c)$.

So we can take $\alpha_i^2 = \frac{2^{19}d^8(c+1)^2}{n}e^{96dc}$, and so $\beta \leq 2^{20}d^8(c+1)^3ne^{96dc}$. Also, for $n \geq \max(1000, 1/\lambda, 64d^2c)$, we can take

$$\alpha = 8d^2e^{48dc} + \frac{512d^4c}{n}e^{48dc} \leq 16d^2e^{48dc}.$$

By Theorem 2.3,

$$\begin{aligned} \mathbb{P}\left(\{|f_{u,v,j,k}(\hat{X}_t) - \mathbb{E}[f_{u,v,j,k}(\hat{X}_t)]| \geq a\} \cap \{X_s \in S_0^0 : 0 \leq s \leq t-1\}\right) \\ \leq 2e^{-a^2/(2^{21}d^8(c+1)^3ne^{96dc}+64d^2e^{48dc}a/3)}, \end{aligned} \quad (5.3)$$

and, similarly,

$$\begin{aligned} \mathbb{P}\left(\{|f_{u,v,\leq j,k}(\hat{X}_t) - \mathbb{E}[f_{u,v,\leq j,k}(\hat{X}_t)]| \geq a\} \cap \{X_s \in S_0^0 : 0 \leq s \leq t-1\}\right) \\ \leq 2e^{-a^2/(2^{21}d^8(c+1)^3ne^{96dc}+64d^2e^{48dc}a/3)}. \end{aligned} \quad (5.4)$$

Now, whenever $\|\hat{X}_{t-1}\|_1 \geq 2\lambda\binom{n}{2}$,

$$\mathbb{P}(\|\hat{X}_t\|_1 - \|\hat{X}_{t-1}\|_1 = 1 | \hat{X}_{t-1}) \leq \frac{\lambda\binom{n}{2}}{\lambda\binom{n}{2} + \lfloor 6\lambda\binom{n}{2} \rfloor} \leq \frac{1}{6},$$

and

$$\mathbb{P}(\|\hat{X}_t\|_1 - \|\hat{X}_{t-1}\|_1 = -1 | \hat{X}_{t-1}) \geq \frac{2\lambda\binom{n}{2}}{\lambda\binom{n}{2} + \lfloor 6\lambda\binom{n}{2} \rfloor} \geq \frac{2}{7}.$$

By, for instance, Lemma 2.5 in [7], it follows that, for each $x_0 \in S_1$,

$$\mathbb{P}(\exists t \leq cn^2 : \hat{X}_t \notin S_0^0) \leq cn^2(7/12)^{2\lambda\binom{n}{2}-1} \leq cn^2e^{-n/4} \leq e^{-n/8}, \quad (5.5)$$

eq:leaveS_0

provided $n \geq \max(1000, 1/\lambda, 64d^2c)$.

Hence, for $n \geq \max(1000, 1/\lambda, 64d^2c)$, $t \leq cn^2$, and $a \leq n$, we have

$$\begin{aligned} \mathbb{P}\left(|f_{v,k}(\hat{X}_t) - \mathbb{E}[f_{v,k}(\hat{X}_t)]| \geq a\right) \\ \leq 2e^{-a^2/(2^{19}d^4(c+1)^3ne^{96dc}+32e^{48dc}a/3)} + e^{-n/8} \\ \leq 4e^{-a^2/2^{20}d^4(c+1)^3ne^{96dc}}. \end{aligned} \quad (5.6)$$

Similarly, we have, for $n \geq \max(1000, 1/\lambda, 64d^2c)$, $t \leq cn^2$, and $a \leq n$,

$$\mathbb{P}\left(|f_{v,\leq k}(\hat{X}_t) - \mathbb{E}[f_{v,\leq k}(\hat{X}_t)]| \geq a\right) \leq 4e^{-a^2/2^{20}d^4(c+1)^3ne^{96dc}} \quad (5.7)$$

$$\mathbb{P}\left(|f_{u,v,j,k}(\hat{X}_t) - \mathbb{E}[f_{u,v,j,k}(\hat{X}_t)]| \geq a\right) \leq 4e^{-a^2/2^{22}d^8(c+1)^3ne^{96dc}} \quad (5.8)$$

$$\mathbb{P}\left(|f_{u,v,\leq j,k}(\hat{X}_t) - \mathbb{E}[f_{u,v,\leq j,k}(\hat{X}_t)]| \geq a\right) \leq 4e^{-a^2/2^{22}d^8(c+1)^3ne^{96dc}} \quad (5.9)$$

We also have

$$\begin{aligned} \mathbb{P}\left(\exists v \in V_n, t \leq cn^2, k \in \{0, \dots, C\}, |f_{v,k}(\hat{X}_t) - \mathbb{E}[f_{v,k}(\hat{X}_t)]| \geq a\right) \\ \leq 4cCn^3e^{-a^2/2^{20}d^4(c+1)^3ne^{96dc}}. \end{aligned} \quad (5.10)$$

A QUANTITATIVE DIFFERENTIAL EQUATION APPROXIMATION

To relate together the continuous-time process X and the discrete-time chain \hat{X} , note that departures in X can be represented by a Poisson process of potential departures with rate $\lfloor 6\lambda \binom{n}{2} \rfloor$, together with a suitable process of ‘choices’ defined as follows. Since X stops as soon as it exits \tilde{S} , while X is moving, at each potential departure we can number all the calls in the system from 1 up to at most $\lfloor 6\lambda \binom{n}{2} \rfloor$. We then draw a number uniformly from $\{1, \dots, \lfloor 6\lambda \binom{n}{2} \rfloor\}$; if the number selected corresponds to a call, that call departs, and otherwise nothing happens. For this representation the mean number of events in X during the interval $[0, t_0]$ is

$$\lambda t_0 \binom{n}{2} + t_0 \lfloor 6\lambda \binom{n}{2} \rfloor \leq 7\lambda t_0 \binom{n}{2},$$

and the events correspond precisely to the jumps of \hat{X} . Let $t_0 = c/8\lambda$, so that the mean number of events in $[0, t_0]$ is at most $\frac{1}{2}cn^2$. Then, by standard concentration inequalities for Poisson random variables, the probability that there are more than cn^2 events of X during $[0, t_0]$ is at most $e^{-cn^2/6}$.

By the above, if $n \geq \max(1000, 1/\lambda, 512d^2\lambda t_0, 1/8\lambda t_0)$, then for $a \leq n$, and any $t \leq t_0$,

$$\begin{aligned} \mathbb{P}\left(|f_{v,k}(X_t) - \mathbb{E}[f_{v,k}(X_t)]| \geq a\right) &\leq 4e^{-a^2/2^{20}d^4(c+1)^3ne^{96dc}} + e^{-cn^2/6} \\ &\leq 4e^{-a^2/2^{20}d^4(8\lambda t_0+1)^3ne^{800d\lambda t_0}} + e^{-n/6} \\ &\leq 8e^{-a^2/2^{20}d^4(8\lambda t_0+1)^3ne^{800d\lambda t_0}}. \end{aligned} \quad (5.11)$$

Also, under the same conditions, we have

$$\mathbb{P}\left(|f_{v,\leq k}(X_t) - \mathbb{E}[f_{v,\leq k}(X_t)]| \geq a\right) \leq 8e^{-a^2/2^{20}d^4(8\lambda t_0+1)^3ne^{800d\lambda t_0}}, \quad (5.12) \quad \boxed{\text{eq. conc-f7a}}$$

$$\mathbb{P}\left(|f_{u,v,j,k}(X_t) - \mathbb{E}[f_{u,v,j,k}(X_t)]| \geq a\right) \leq 8e^{-a^2/2^{22}d^8(8\lambda t_0+1)^3ne^{800d\lambda t_0}}, \quad (5.13) \quad \boxed{\text{eq. conc-f8}}$$

$$\mathbb{P}\left(|f_{u,v,\leq j,k}(X_t) - \mathbb{E}[f_{u,v,\leq j,k}(X_t)]| \geq a\right) \leq 8e^{-a^2/2^{22}d^8(8\lambda t_0+1)^3ne^{800d\lambda t_0}}. \quad (5.14) \quad \boxed{\text{eq. conc-f8a}}$$

The above inequalities are all uniform in $d \geq 1$, $u, v \in V_n$, and $j, k \in \{0, 1, \dots, C\}$. In particular, if we set

$$\gamma = \frac{1}{2^{24}d^8(8\lambda t_0+1)^3e^{800d\lambda t_0}},$$

and take $a = \frac{1}{2}\sqrt{n} \log n$, then we obtain that, for $n \geq \max(1000, 1/\lambda, 512d^2\lambda t_0, 1/8\lambda t_0)$, $t \leq t_0$, $d \geq 1$, and each u, v and j, k ,

$$\mathbb{P}\left(|f_{v,k}(X_t) - \mathbb{E}[f_{v,k}(X_t)]| \geq \frac{1}{2}\sqrt{n} \log n\right) \leq 8e^{-\gamma \log^2 n}. \quad (5.15)$$

$$\mathbb{P}\left(|f_{v,\leq k}(X_t) - \mathbb{E}[f_{v,\leq k}(X_t)]| \geq \frac{1}{2}\sqrt{n} \log n\right) \leq 8e^{-\gamma \log^2 n}. \quad (5.16)$$

$$\mathbb{P}\left(|f_{u,v,j,k}(X_t) - \mathbb{E}[f_{u,v,j,k}(X_t)]| \geq \frac{1}{2}\sqrt{n} \log n\right) \leq 8e^{-\gamma \log^2 n} \quad (5.17)$$

$$\mathbb{P}\left(|f_{u,v,\leq j,k}(X_t) - \mathbb{E}[f_{u,v,\leq j,k}(X_t)]| \geq \frac{1}{2}\sqrt{n \log n}\right) \leq 8e^{-\gamma \log^2 n}. \quad (5.18)$$

Hence

$$\begin{aligned} \mathbb{P}\left(\exists v \in V_n, t \leq cn^2, k \in \{0, \dots, C\}, |f_{v,k}(X_t) - \mathbb{E}[f_{v,k}(X_t)]| \geq \frac{1}{2}\sqrt{n \log n}\right) \\ \leq 8cCn^3 e^{-\gamma \log^2 n}. \end{aligned} \quad (5.19)$$

6. EXPECTATION OF THE GENERATOR

sec:gen-expec

As before, we assume that our process starts in some fixed state $x_0 \in S_1$, and we consider the law of the process started in this state, and running until some time $t_0 > 0$.

Recall that

$$\gamma = \gamma(\lambda, d, t_0) = \frac{1}{2^{24}d^8(8\lambda t_0 + 1)^3 e^{800d\lambda t_0}},$$

and

$$n_0(\lambda, d, C, t_0) = \max\left(\frac{1}{\lambda}, \frac{1}{8\lambda t_0}, 10^7(\lambda + 1)^4 d^4 (C + 1)^6 t_0^2, e^{8/\gamma(\lambda, d, t_0)}\right).$$

Note that $n_0(\lambda, d, C, t_0) \geq e^{2^{25}} \geq 1000$, for any positive integers d and C and positive reals λ and t_0 , so that the bounds of the previous section hold for all $n \geq n_0(\lambda, d, C, t_0)$. We also have $e^{-\gamma \log^2 n} \leq n^{-8}$, an inequality we shall use freely from now on.

Recall the definitions of ϕ^1 , ϕ^2 and ϕ^3 from (1.4)–(1.6), and that $\phi = \max\{\phi^1, \phi^2, \phi^3\}$. Set $\tilde{\phi} = \max\{\phi^1, \phi^2\}$. Recall also that

$$\begin{aligned} g_j(\xi) &= 2\xi(C)\xi(j)\xi(\leq j) \sum_{r=1}^d (1 - \xi(\leq j)^2)^{r-1} (1 - \xi(\leq j-1)^2)^{d-r} \\ &\quad + 2\xi(C)\xi(j) \sum_{i=j+1}^{C-1} \xi(i) \sum_{r=1}^d (1 - \xi(\leq i)^2)^{r-1} (1 - \xi(\leq i-1)^2)^{d-r}. \end{aligned}$$

Our first aim in this section is to show that, provided $\mathbb{E}\phi(X_t)$ is small, $\mathbb{E}[g_{v,j}(X_t)]$ is close to $(n-1)g_j(z_t^v)$, where z_t^v is the vector with components $z_t(v, j) = (n-1)^{-1} \mathbb{E}[f_{v,j}(X_t)]$, for $j \in \{0, \dots, C\}$. We then go on to show that, if $\phi(x_0)$ is small, then also $\mathbb{E}\phi(X_t)$ is small for all $t \leq t_0$.

lem.gen-expec

Lemma 6.1. *For all $t \leq t_0$, for each $v \in V_n$ and each $j \in \{0, \dots, C\}$,*

$$\begin{aligned} &|\mathbb{E}[g_{v,j}(X_t)] - (n-1)g_j(z_t^v)| \\ &\leq 22d^2(C+1)^3 n \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}}\right) e^{208(\lambda+1)d^2(C+1)^3 t_0}, \end{aligned}$$

provided $n \geq n_0(\lambda, d, C, t_0)$, where z_t^v is the vector with components $z_t(v, j)$ ($j \in \{0, \dots, C\}$), with $z_t(v, j) = (n-1)^{-1} \mathbb{E}[f_{v,j}(X_t)]$.

If $d = 1$ and $n \geq n_0(\lambda, 1, C, t_0)$, we have the improved bound

$$|\mathbb{E}[g_{v,j}(X_t)] - (n-1)g_j(z_t^v)|$$

$$\leq 22(C+1)^3 n \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) e^{208(\lambda+1)(C+1)t_0}.$$

The lemma above will follow immediately from two other lemmas, the first of which is as follows.

lem.gen-exp

Lemma 6.2. *For any $n \geq n_0(\lambda, d, C, t_0)$, $v \in V_n$ and $j \in \{0, \dots, C\}$,*

$$\left| \mathbb{E}[g_{v,j}(X_t)] - (n-1)g_j(z_t^v) \right| \leq 8d^2(C+1)^3 n \mathbb{E}[\tilde{\phi}(X_t)] + 16d^2(C+1)\sqrt{n} \log n.$$

Proof. Suppose that $n \geq n_0(\lambda, d, C, t_0)$.

The function $g_{v,j}$ introduced earlier (3.1) is a sum of four terms. We start by separating out these terms. Let

$$\begin{aligned} P_{v,j}^+ &= \frac{1}{(n-2)^d} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^C \sum_{w_r} \mathbb{I}_{vw_r}^j \mathbb{I}_{uw_r}^{\leq j} \prod_{s=1}^{r-1} \sum_{w_s} (1 - \mathbb{I}_{uv,w_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s} (1 - \mathbb{I}_{uv,w_s}^{\leq j-1}); \\ P_{v,j}^- &= \frac{1}{(n-2)^d} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^C \sum_{w_r} \mathbb{I}_{vw_r}^j \sum_{i=j+1}^{C-1} \mathbb{I}_{uw_r}^i \prod_{s=1}^{r-1} \sum_{w_s} (1 - \mathbb{I}_{uv,w_s}^{\leq i}) \\ &\quad \times \prod_{s=r+1}^d \sum_{w_s} (1 - \mathbb{I}_{uv,w_s}^{\leq i-1}). \end{aligned}$$

In both expressions above, the first sum is over all values of $u \neq v$, and the subsequent sums are over all values of w_r or $w_s \neq u, v$. Let further

$$\begin{aligned} Q_{v,j}^+ &= \frac{1}{(n-2)^d} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^j \sum_{v'} \mathbb{I}_{uv'}^C \mathbb{I}_{v'v}^{\leq j} \prod_{s=1}^{r-1} \sum_{w_s} (1 - \mathbb{I}_{uv',w_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s} (1 - \mathbb{I}_{uv',w_s}^{\leq j-1}); \\ Q_{v,j}^- &= \frac{1}{(n-2)^d} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^j \sum_{v'} \mathbb{I}_{uv'}^C \sum_{i=j+1}^{C-1} \mathbb{I}_{v'v}^i \prod_{s=1}^{r-1} \sum_{w_s} (1 - \mathbb{I}_{uv',w_s}^{\leq i}) \\ &\quad \times \prod_{s=r+1}^d \sum_{w_s} (1 - \mathbb{I}_{uv',w_s}^{\leq i-1}). \end{aligned}$$

In the last two expressions above, the first sum is over all values of $u \neq v$, the second sum is over all values of $v' \neq u, v$, and the subsequent sums are over all $w_s \neq u, v'$.

Then

$$g_{v,j} = P_{v,j}^+ + P_{v,j}^- + Q_{v,j}^+ + Q_{v,j}^-.$$

We define further ‘standardised’ versions of $P_{v,j}^+$, $P_{v,j}^-$, $Q_{v,j}^+$, $Q_{v,j}^-$. Let

$$\hat{P}_{v,j}^+ = \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^C \sum_{w_r} \mathbb{I}_{vw_r}^j \sum_{w'_r} \mathbb{I}_{uw'_r}^{\leq j}$$

$$\begin{aligned}
& \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}); \\
\hat{P}_{v,j}^- &= \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^C \sum_{w_r} \mathbb{I}_{vw_r}^j \sum_{i=j+1}^{C-1} \sum_{w'_r} \mathbb{I}_{uw'_r}^i \\
& \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq i} \mathbb{I}_{vw'_s}^{\leq i}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq i-1} \mathbb{I}_{vw'_s}^{\leq i-1}).
\end{aligned}$$

In these two expressions, the first sum is over all values of $u \neq v$, and the remaining sums are over all values of $w_r, w'_r, w_s, w'_s \neq u, v$. Also, let

$$\begin{aligned}
\hat{Q}_{v,j}^+ &= \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^j \sum_{v'} \mathbb{I}_{uv'}^C \sum_{v''} \mathbb{I}_{v''v}^{\leq j} \\
& \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}); \\
\hat{Q}_{v,j}^- &= \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^j \sum_{v'} \mathbb{I}_{uv'}^C \sum_{i=j+1}^{C-1} \sum_{v''} \mathbb{I}_{v''v}^i \\
& \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq i} \mathbb{I}_{vw'_s}^{\leq i}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq i-1} \mathbb{I}_{vw'_s}^{\leq i-1}).
\end{aligned}$$

In these two expressions, the first sum is over all values of $u \neq v$, and the subsequent sums are over all values of $v', v'', w_s, w'_s \neq u, v$. In these standardised versions, the “anchor” nodes u and v for the final products are chosen so that the products can be extracted as far as possible as a common factor. In the future, we will always use similar conventions regarding the ranges of the various summations involved, and the choices of anchor nodes.

Set

$$\hat{g}_{v,j} = \hat{P}_{v,j}^+ + \hat{P}_{v,j}^- + \hat{Q}_{v,j}^+ + \hat{Q}_{v,j}^-.$$

We shall now bound $|g_{v,j} - \hat{g}_{v,j}|$ above via upper bounds on the differences $|P_{v,j}^+ - \hat{P}_{v,j}^+|$, $|P_{v,j}^- - \hat{P}_{v,j}^-|$, $|Q_{v,j}^+ - \hat{Q}_{v,j}^+|$ and $|Q_{v,j}^- - \hat{Q}_{v,j}^-|$. These differences denote the maximum difference of the functions over all load vectors x .

Noting that, for $0 \leq j \leq C$ and for any u ,

$$\begin{aligned}
& \left| \frac{1}{n-2} \sum_{w_s} \left(1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw_s}^{\leq j}\right) - \frac{1}{(n-2)^2} \sum_{w_s, w'_s} \left(1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}\right) \right| \\
&= \left| \frac{1}{n-2} \sum_{w_s} \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw_s}^{\leq j} - \frac{1}{(n-2)^2} \sum_{w_s, w'_s} \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j} \right| \leq (C+1)^2 \phi^1,
\end{aligned}$$

we see that

$$|P_{v,j}^+ - \hat{P}_{v,j}^+| \leq d^2(C+1)^2(n-2)\phi^1.$$

Similarly,

$$|P_{v,j}^- - \hat{P}_{v,j}^-| \leq d^2(C+1)^3(n-2)\phi^1.$$

Also,

$$\begin{aligned} & |Q_{v,j}^+ - \hat{Q}_{v,j}^+| \\ & \leq d(d-1)[(C+1)^2(n-2)\phi^1 + (C+1)(n-2)\phi^2] \\ & \quad + \frac{1}{(n-2)^{2d-1}} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^j \left| \sum_{v'} \mathbb{I}_{uv'}^C \left(\mathbb{I}_{v'v}^{\leq j} - \frac{1}{n-2} \sum_{v''} \mathbb{I}_{v''v}^{\leq j} \right) \right. \\ & \quad \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}) \Big| \\ & \leq d(d-1)(C+1)^2(n-2)(\phi^1 + \phi^2) + \frac{1}{(n-2)^{2d-2}} \sum_{r=1}^d \sum_u \mathbb{I}_{uv}^j \\ & \quad \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}) \\ & \quad \times \left| \frac{1}{n-2} \sum_{v'} \mathbb{I}_{uv'}^C \mathbb{I}_{v'v}^{\leq j} - \frac{1}{(n-2)^2} \sum_{v'} \mathbb{I}_{uv'}^C \sum_{v''} \mathbb{I}_{v''v}^{\leq j} \right|. \end{aligned}$$

Hence

$$\begin{aligned} |Q_{v,j}^+ - \hat{Q}_{v,j}^+| & \leq d(d-1)(C+1)^2(n-2)(\phi^1 + \phi^2) + d(C+1)(n-2)\phi^1 \\ & \leq 2d^2(C+1)^2(n-2)\tilde{\phi}. \end{aligned}$$

Similarly, $|Q_{v,j}^- - \hat{Q}_{v,j}^-| \leq 2d^2(C+1)^3(n-2)\tilde{\phi}$.

It follows that

$$|g_{v,j} - \hat{g}_{v,j}| \leq 6d^2(C+1)^3(n-2)\tilde{\phi}, \quad (6.1) \quad \boxed{\text{gghat1}}$$

and so

$$|\mathbb{E}[g_{v,j}(X_t)] - \mathbb{E}[\hat{g}_{v,j}(X_t)]| \leq 6d^2(C+1)^3(n-2)\mathbb{E}[\tilde{\phi}(X_t)].$$

In the special case $d = 1$, the estimates are easier, and we find that

$$|g_{v,j} - \hat{g}_{v,j}| \leq 4(C+1)(n-2)\tilde{\phi}. \quad (6.2) \quad \boxed{\text{gghat1}}$$

Our next task is to estimate $|\mathbb{E}[\hat{g}_{v,j}(X_t)] - (n-1)g_j(z_t^v)|$, where $(n-1)g_j(z_t^v)$ is given by

$$\begin{aligned} & \frac{2}{(n-1)^{d+1}} \mathbb{E}[f_{v,C}] \mathbb{E}[f_{v,j}] \mathbb{E}[f_{v,\leq j}] \sum_{r=1}^d ((n-1) - \mathbb{E}[f_{v,\leq j}]^2)^{r-1} \\ & \quad \times ((n-1) - \mathbb{E}[f_{v,\leq j-1}]^2)^{d-r} \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{(n-1)^{d+1}} \mathbb{E}[f_{v,C}] \mathbb{E}[f_{v,j}] \sum_{i=j+1}^{C-1} \mathbb{E}[f_{v,\leq i}] \sum_{r=1}^d ((n-1) - \mathbb{E}[f_{v,\leq i}]^2)^{r-1} \\
& \quad \times ((n-1) - \mathbb{E}[f_{v,\leq i-1}]^2)^{d-r}.
\end{aligned}$$

Here, and throughout what follows, we abuse notation by writing e.g. $\mathbb{E}[f_{v,C}]$ instead of $\mathbb{E}[f_{v,C}(X_t)]$: for the remainder of this proof, all of our functions will be evaluated at X_t .

We start by estimating the difference between $\mathbb{E}\hat{P}_{v,j}^+$ and

$$\frac{\mathbb{E}[f_{v,C}] \mathbb{E}[f_{v,j}] \mathbb{E}[f_{v,\leq j}]}{(n-1)^{d+1}} \sum_{r=1}^d ((n-1) - \mathbb{E}[f_{v,\leq j}]^2)^{r-1} ((n-1) - \mathbb{E}[f_{v,\leq j-1}]^2)^{d-r} :$$

$|\mathbb{E}[\hat{g}_{v,j}(X_t)] - (n-1)g_j(z_t^v)|$ is the sum of this and three similar terms.

Note that

$$\hat{P}_{v,j}^+ = \frac{1}{n-2} \sum_{u \neq v} \mathbb{I}_{u,v}^C f_{u,v,\leq j,j}.$$

By (5.18), as $n \geq n_0(\lambda, d, C, t_0)$, the function

$$\begin{aligned}
f_{u,v,\leq j,j} &= \frac{1}{(n-2)^{2d-1}} \sum_{r=1}^d \sum_{w_r} \mathbb{I}_{vw_r}^j \sum_{w'_r} \mathbb{I}_{uw'_r}^{\leq j} \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \\
& \quad \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1})
\end{aligned}$$

satisfies, for each $t \leq t_0$,

$$\mathbb{P} \left(|f_{u,v,\leq j,j}(X_t) - \mathbb{E}[f_{u,v,\leq j,j}(X_t)]| \geq \frac{1}{2} \sqrt{n} \log n \right) \leq 8e^{-\gamma \log^2 n}.$$

By Lemma 2.5, as $n \geq n_0(\lambda, d, C, t_0)$,

$$\left| \mathbb{E}[\mathbb{I}_{uv}^C f_{u,v,\leq j,j}] - \mathbb{E}[\mathbb{I}_{uv}^C] \mathbb{E}[f_{u,v,\leq j,j}] \right| \leq \frac{1}{2} \sqrt{n} \log n + 8ne^{-\gamma \log^2 n} \leq \sqrt{n} \log n,$$

for each $u \neq v$, and so

$$\left| \mathbb{E}[\hat{P}_{v,j}^+] - \frac{1}{n-2} \sum_u \mathbb{E}[\mathbb{I}_{uv}^C] \mathbb{E}[f_{u,v,\leq j,j}] \right| \leq \frac{n-1}{n-2} \sqrt{n} \log n \leq 2\sqrt{n} \log n. \quad (6.3) \quad \boxed{\text{hatP+}}$$

Now let E_t be the event that $|f_{v,j}(X_t) - \mathbb{E} f_{v,j}(X_t)| \leq \frac{1}{2} \sqrt{n} \log n$ and $|f_{v,\leq j}(X_t) - \mathbb{E} f_{v,\leq j}(X_t)| \leq \frac{1}{2} \sqrt{n} \log n$ for all $j \in \{0, \dots, C\}$ and $v \in V_n$. By (5.15) and (5.16), $\mathbb{P}(\overline{E_t}) \leq 16(C+1)ne^{-\gamma \log^2 n}$.

Note that, on E_t ,

$$\frac{1}{n-2} \left| (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j})(f_{v,\leq j} - \mathbb{I}_{uv}^{\leq j}) - \mathbb{E}[f_{u,\leq j}] \mathbb{E}[f_{v,\leq j}] \right| \leq \frac{3}{2} \sqrt{n} \log n;$$

$$\frac{1}{n-2} \left| (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j})(f_{v,j} - \mathbb{I}_{uv}^j) - \mathbb{E}[f_{u,\leq j}] \mathbb{E}[f_{v,j}] \right| \leq \frac{3}{2} \sqrt{n} \log n,$$

for each j . Thus, recalling that

$$\begin{aligned} f_{u,v,\leq j,j} &= \frac{1}{(n-2)^{2d-1}} (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j})(f_{v,j} - \mathbb{I}_{uv}^j) \\ &\quad \times \sum_{r=1}^d \left((n-2)^2 - (f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j})(f_{v,\leq j} - \mathbb{I}_{uv}^{\leq j}) \right)^{r-1} \\ &\quad \times \left((n-2)^2 - (f_{u,\leq j-1} - \mathbb{I}_{uv}^{\leq j-1})(f_{v,\leq j-1} - \mathbb{I}_{uv}^{\leq j-1}) \right)^{d-r}, \end{aligned}$$

we see that, on E_t , the difference between $f_{u,v,\leq j,j}$ and

$$\begin{aligned} &\frac{1}{(n-2)^{2d-1}} \mathbb{E}[f_{u,\leq j}] \mathbb{E}[f_{v,j}] \sum_{r=1}^d \left((n-2)^2 - \mathbb{E}[f_{u,\leq j}] \mathbb{E}[f_{v,\leq j}] \right)^{r-1} \\ &\quad \times \left((n-2)^2 - \mathbb{E}[f_{u,\leq j-1}] \mathbb{E}[f_{v,\leq j-1}] \right)^{d-r} \end{aligned}$$

is at most $\frac{3}{2}d^2\sqrt{n}\log n$ in absolute value.

Thus, with probability at least $1 - 32(C+1)ne^{-\gamma\log^2 n}$, $f_{u,v,\leq j,j}$ is within distance $\frac{1}{2}\sqrt{n}\log n$ of $\mathbb{E}[f_{u,v,\leq j,j}]$ and within distance $\frac{3}{2}d^2\sqrt{n}\log n$ of

$$\begin{aligned} H_{u,v,j}(t) &= \frac{1}{(n-2)^{2d-1}} \mathbb{E}[f_{u,\leq j}] \mathbb{E}[f_{v,j}] \sum_{r=1}^d \left((n-2)^2 - \mathbb{E}[f_{u,\leq j}] \mathbb{E}[f_{v,\leq j}] \right)^{r-1} \\ &\quad \times \left((n-2)^2 - \mathbb{E}[f_{u,\leq j-1}] \mathbb{E}[f_{v,\leq j-1}] \right)^{d-r}. \end{aligned}$$

It follows that the difference between $\mathbb{E}[f_{u,v,\leq j,j}]$ and $H_{u,v,j}(t)$ is at most $2d^2\sqrt{n}\log n$. Since, for each u, v and j , $|\mathbb{E}[f_{u,\leq j}] - \mathbb{E}[f_{v,\leq j}]| \leq (n-2)(C+1)\mathbb{E}[\phi^2]$, the difference between $\mathbb{E}[f_{u,v,\leq j,j}]$ and $H_{u,v,j}(t)$ is at most $2d^2\sqrt{n}\log n + d^2(n-2)(C+1)\mathbb{E}[\phi^2]$, for each u, v and j .

Combining the above with (6.3), we see that the difference between $\mathbb{E}[\hat{P}_{v,j}^+]$ and $H_{u,v,j}(t)$ is at most $4d^2\sqrt{n}\log n + d^2n(C+1)\mathbb{E}[\phi^2]$ in absolute value.

A similar argument shows that the difference between $\mathbb{E}[\hat{P}_{v,j}^-]$ and

$$\begin{aligned} &\frac{1}{(n-2)^{2d}} \mathbb{E}[f_{v,C}] \mathbb{E}[f_{v,j}] \sum_{i=j+1}^{C-1} \mathbb{E}[f_{v,i}] \sum_{r=1}^d \left((n-2)^2 - (\mathbb{E}[f_{v,\leq i}])^2 \right)^{r-1} \\ &\quad \times \left((n-2)^2 - (\mathbb{E}[f_{v,\leq i-1}])^2 \right)^{d-r} \end{aligned}$$

is at most $4d^2C\sqrt{n}\log n + d^2nC(C+1)\mathbb{E}[\phi^2]$ in absolute value.

For $\mathbb{E}[\hat{Q}_{v,j}^+]$, we use an argument identical to the one above, considering

$$f_{v,u,\leq j,C} = \frac{1}{(n-2)^{2d-1}} \sum_{r=1}^d \sum_{v'} \mathbb{I}_{uv'}^C \sum_{v''} \mathbb{I}_{v''v}^{\leq j}$$

$$\times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1})$$

to show that the difference between $\mathbb{E}[\hat{Q}_{v,j}^+]$ and

$$\frac{1}{(n-2)^{2d}} \mathbb{E}[f_{v,C}] \mathbb{E}[f_{v,\leq j}] \mathbb{E}[f_{v,j}] \sum_{r=1}^d \left((n-2)^2 - (\mathbb{E}[f_{v,\leq j}])^2 \right)^{r-1} \times \left((n-2)^2 - (\mathbb{E}[f_{v,\leq j-1}])^2 \right)^{d-r}$$

is at most $4d^2 \sqrt{n} \log n + d^2 n(C+1) \mathbb{E}[\phi^2]$ in absolute value.

Similarly, the difference between $\mathbb{E}[\hat{Q}_{v,j}^-]$ and

$$\frac{1}{(n-2)^{2d}} \mathbb{E}[f_{v,C}] \mathbb{E}[f_{v,j}] \sum_{i=j+1}^{C-1} \mathbb{E}[f_{v,i}] \sum_{r=1}^d \left((n-2)^2 - (\mathbb{E}[f_{v,\leq i}])^2 \right)^{r-1} \times \left((n-2)^2 - (\mathbb{E}[f_{v,\leq i-1}])^2 \right)^{d-r}$$

is at most $4d^2 C \sqrt{n} \log n + d^2 nC(C+1) \mathbb{E}[\phi^2]$ in absolute value.

In summary, we have shown that

$$\begin{aligned} & \left| \mathbb{E}[g_{v,j}(X_t)] - (n-2)g_j(y_t^v) \right| \\ & \leq \left| \mathbb{E}[g_{v,j}(X_t)] - \mathbb{E}[\hat{g}_{v,j}(X_t)] \right| + \left| \mathbb{E}[\hat{g}_{v,j}(X_t)] - (n-2)g_j(y_t^v) \right| \\ & \leq 8d^2(C+1)^3 n \mathbb{E}[\tilde{\phi}(X_t)] + 8d^2(C+1) \sqrt{n} \log n, \end{aligned}$$

where y_t^v is the vector with components $y_t(v, j) = \frac{1}{n-2} \mathbb{E} f_{v,j}(X_t)$.

Now, each of the components $z_t(v, j)$ and $y_t(v, j)$ is non-negative, and we have $\sum_{j=0}^C z_t(v, j) \leq 1$ and $\sum_{j=0}^C y_t(v, j) \leq \frac{n-1}{n-2}$. Furthermore, $|z_t(v, j) - y_t(v, j)| \leq \frac{1}{n-2}$ for all j . Also, exactly as in the proof of (7.1) below, whenever y and z are in $\{x \in \mathbb{R}^{C+1} : x(j) \geq 0 \text{ for each } j, \sum_j x(j) \leq \frac{n-1}{n-2}\}$, we have

$$|g_k(y) - g_k(z)| \leq 3d^2(C+1)^2 \left(\frac{n-1}{n-2} \right)^3 \max_{0 \leq j \leq C} |y(j) - z(j)|.$$

It follows that, for $n \geq 6$,

$$|g_j(y_t^v) - g_j(z_t^v)| \leq 3d^2(C+1)^2 \left(\frac{n-1}{n-2} \right)^3 \frac{1}{n-2} \leq 6d^2(C+1)^2 \frac{1}{n-2},$$

and so, using the fact that $|g_j(x)| \leq 2d(C+1)$ for any load vector x ,

$$\begin{aligned} & \left| \mathbb{E}[g_{v,j}(X_t)] - (n-1)g_j(z_t^v) \right| \\ & \leq 8d^2(C+1)^3 n \mathbb{E}[\tilde{\phi}(X_t)] + 8d^2(C+1) \sqrt{n} \log n + 6d^2(C+1)^2 + 2d(C+1). \end{aligned}$$

As $n \geq (C+1)^2$, we may now write

$$\left| \mathbb{E}[g_{v,j}(X_t)] - (n-1)g_j(z_t^v) \right| \leq 8d^2(C+1)^3 n \mathbb{E}[\tilde{\phi}(X_t)] + 16d^2(C+1) \sqrt{n} \log n,$$

as claimed. \square

We now need to study the changes of $\phi(X_t)$ over time.

For distinct vertices u and v , and $j, k \in \{0, \dots, C\}$, we define

$$\begin{aligned}\phi_{u,v,j,k}^1 &= \frac{1}{n-2} \sum_w \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k - \frac{1}{(n-2)^2} \sum_{w \neq u,v} \mathbb{I}_{uw}^j \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k; \\ \phi_{u,v,j}^2 &= \frac{1}{n-2} (f_{u,j} - f_{v,j}) = \frac{1}{n-2} \left(\sum_{w \neq u,v} \mathbb{I}_{uw}^j - \sum_{w \neq u,v} \mathbb{I}_{vw}^j \right); \\ \phi_{u,v}^3(x) &= \frac{1}{n-2} \sum_{w \neq u,v} x(\{u, v\}, w).\end{aligned}$$

Then we have $\phi^1 = \max_{u,v,j,k} |\phi_{u,v,j,k}^1|$, $\phi^2 = \max_{u,v,j} |\phi_{u,v,j}^2|$ and $\phi^3 = \max_{u,v} \phi_{u,v}^3$, where all maximisations are over distinct vertices u and v and, where appropriate, $j, k \in \{0, \dots, C\}$. These functions are similar to ones in [3]: we prove an analogue of Lemma 2 in [3], leaving some details to an appendix, but our task is more complex as we deal with $d > 1$, and we fill in a key point that is dealt with rather brusquely in [3].

Once again, our argument uses the discrete chain (\hat{X}_t) . Let $(\hat{\mathcal{F}}_t)$ denote the natural filtration of (\hat{X}_t) . Let $B_{t-1} = \{\hat{X}_s \in \tilde{S} \text{ for all } s \leq t-1\}$. For a function $f : S \rightarrow \mathbb{R}$, we define $\Delta f(\hat{X}_t) = f(\hat{X}_t) - f(\hat{X}_{t-1})$, the increment of the function on one step of the discrete chain. Our first goal is to provide upper bounds on $\mathbb{E}[|\Delta \phi_{u,v,j,k}^1(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}]$, $\mathbb{E}[|\Delta \phi_{u,v,j}^2(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}]$ and $\mathbb{E}[|\Delta \phi_{u,v}^3(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}]$, in terms of $\phi(\hat{X}_{t-1})$, valid, on the event B_{t-1} , for all distinct nodes u and v , and, where appropriate, all $j, k \in \{0, \dots, C\}$.

The proof of the following lemma consists of routine but tedious calculations, and these are relegated to the appendix.

lem.appendix

Lemma 6.3. *Suppose $t \leq t_0$ and $n \geq n_0(\lambda, d, C, t_0)$. For ρ any one of the functions $\phi_{u,v,j,k}^1$, $\phi_{u,v,j}^2$, or $\phi_{u,v}^3$, we have, on B_{t-1} ,*

$$\mathbb{E}[|\Delta \rho(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \leq \frac{c_1}{n^2} \phi(\hat{X}_{t-1}) + \frac{c_2}{n^3},$$

where $c_1 = 26(1 + 1/\lambda)d^2(C+1)^3$ and $c_2 = 64\lambda d^2(C+1)^3$.

If $d = 1$, we have the same conclusion with $c_1 = 26(1 + 1/\lambda)(C+1)$ and $c_2 = 64\lambda(C+1)$.

Now we are in a position to prove the other result required for Lemma 6.1.

lem.phi-over-time

Lemma 6.4. *For all $t \leq t_0$, and $n \geq n_0(\lambda, d, C, t_0)$, we have*

$$\mathbb{E} \phi(X_t) \leq 2e^{208(\lambda+1)d^2(C+1)^3 t_0} \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right).$$

If $d = 1$, and $n \geq n_0(\lambda, 1, C, t_0)$, we have the improved bound

$$\mathbb{E} \phi(X_t) \leq 2e^{208(\lambda+1)(C+1)t_0} \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right).$$

Proof. Let $m = \phi(x_0) + \frac{3\log n}{\sqrt{n}}$, and let C_t be the event

$$B_t \cap \left\{ \phi(\hat{X}_s) \leq m \left(1 + \frac{c_1}{n^2} \right)^s, \text{ for all } s \leq t \right\}.$$

Let ρ denote any of the functions $\phi_{u,v,j,k}^1$, $\phi_{u,v,j}^2$ or $\phi_{u,v}^3$. For each t , on the event C_{t-1} , we have from Lemma 6.3 that

$$\mathbb{E}(|\rho(\hat{X}_t)| - |\rho(\hat{X}_{t-1})| \mid \hat{\mathcal{F}}_{t-1}) \leq \mathbb{E}(|\Delta\rho(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}) \leq \frac{c_1}{n^2} m \left(1 + \frac{c_1}{n^2} \right)^{t-1} + \frac{c_2}{n^3},$$

and therefore

$$\mathbb{E}(|\rho(\hat{X}_t)| \mid \mathbb{I}_{C_{t-1}}) \leq \mathbb{E}(|\rho(\hat{X}_{t-1})| \mid \mathbb{I}_{C_{t-2}}) + \frac{c_1}{n^2} m \left(1 + \frac{c_1}{n^2} \right)^{t-1} + \frac{c_2}{n^3}.$$

This yields, for each t ,

$$\begin{aligned} \mathbb{E}(|\rho(\hat{X}_t)| \mid \mathbb{I}_{C_{t-1}}) &\leq |\rho(x_0)| + \sum_{s=0}^{t-1} \left(\frac{c_1}{n^2} m \left(1 + \frac{c_1}{n^2} \right)^{s-1} + \frac{c_2}{n^3} \right) \\ &\leq \phi(x_0) + m \left\{ \left(1 + \frac{c_1}{n^2} \right)^t - 1 \right\} + \frac{c_2 t}{n^3} \\ &= m \left(1 + \frac{c_1}{n^2} \right)^t - \frac{3\log n}{\sqrt{n}} + \frac{c_2 t}{n^3}. \end{aligned}$$

Now let $c = 8\lambda t_0$, and run the discrete chain for cn^2 steps. Note that

$$n \geq n_0(\lambda, d, C, t_0) \geq 10^7(\lambda + 1)^4 d^4 (C + 1)^6 t_0^2 \geq (cc_2)^2.$$

For $t \leq cn^2$, we conclude that

$$\mathbb{E}(|\rho(\hat{X}_t)| \mid \mathbb{I}_{C_{t-1}}) \leq m \left(1 + \frac{c_1}{n^2} \right)^t - \frac{2\log n}{\sqrt{n}}.$$

We now show by induction on t that $\mathbb{P}(\overline{C_t}) \leq te^{-\frac{1}{2}\gamma \log^2 n}$, for all $t \leq cn^2$. This is certainly true for $t = 0$. If the induction hypothesis holds for $t - 1$, then we have

$$\mathbb{E}(|\rho(\hat{X}_t)| \mid \overline{C_{t-1}}) \leq C \mathbb{P}(\overline{C_{t-1}}) \leq Ccn^2 e^{-\frac{1}{2}\gamma \log^2 n} \leq \frac{\log n}{\sqrt{n}},$$

and so

$$\mathbb{E}(|\rho(\hat{X}_t)|) \leq m \left(1 + \frac{c_1}{n^2} \right)^t - \frac{\log n}{\sqrt{n}}.$$

Thus

$$\mathbb{P}(\overline{C_t}) \leq \mathbb{P}(\overline{B_t}) + \mathbb{P}(\overline{C_{t-1}}) + (C + 2)^2 n^2 \max_{\rho} \mathbb{P}(|\rho(\hat{X}_t)| \geq \mathbb{E}|\rho(\hat{X}_t)| + \frac{\log n}{\sqrt{n}}),$$

where the maximum is over all the functions $\phi_{u,v,k,j}^1$, $\phi_{u,v,j}^2$ and $\phi_{u,v}^3$, noting that there are at most $(C + 2)^2 n^2$ such functions.

Inequality (5.5) implies that

$$\mathbb{P}(\overline{B_t}) \leq e^{-n/8} \leq \frac{1}{2} e^{-\frac{1}{2}\gamma \log^2 n},$$

for $t \leq cn^2$, since the chain starts in S_1 .

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To establish concentration of measure for the functions $|\phi_{u,v,j,k}^1|$, $|\phi_{u,v,j}^2|$ and $\phi_{u,v}^3$, we proceed as in Sections 4 and 5. Indeed, it is easy to see that

$$(n-2) \left| |\phi_{u,v,j,k}^1(x)| - |\phi_{u,v,j,k}^1(y)| \right| \leq 2(\|x-y\|_u + \|x-y\|_v),$$

$$(n-2) \left| |\phi_{u,v,j}^2(x)| - |\phi_{u,v,j}^2(y)| \right| \leq 2(\|x-y\|_u + \|x-y\|_v),$$

and

$$(n-2) \left| \phi_{u,v}^3(x) - \phi_{u,v}^3(y) \right| \leq \|x-y\|_u,$$

for all u, v, j and k , and all x and y in \tilde{S} . Calculations exactly as leading up to (5.6) now give, for any of the functions ρ , any $t \leq cn^2$, and any $a \leq n$,

$$\mathbb{P} \left(\left| |\rho(\hat{X}_t)| - \mathbb{E} |\rho(\hat{X}_t)| \right| \geq \frac{a}{n-2} \right) \leq 4e^{-a^2/2^{22}d^4(c+1)^3ne^{96dc}} \leq 4e^{-4\gamma a^2/n}.$$

Applying this with $a = \frac{1}{2}\sqrt{n} \log n$ gives

$$\mathbb{P} \left(\left| |\rho(\hat{X}_t)| - \mathbb{E} |\rho(\hat{X}_t)| \right| \geq \frac{\log n}{\sqrt{n}} \right) \leq 4e^{-\gamma \log^2 n} \leq 4n^{-4}e^{-\frac{1}{2}\gamma \log^2 n}.$$

We thus have, using also the induction hypothesis, that

$$\mathbb{P}(\overline{C}_t) \leq \frac{1}{2}e^{-\frac{1}{2}\gamma \log^2 n} + (t-1)e^{-\frac{1}{2}\gamma \log^2 n} + 4(C+2)^2n^2n^{-4}e^{-\frac{1}{2}\gamma \log^2 n} \leq te^{-\frac{1}{2}\gamma \log^2 n},$$

as required for the induction step.

This proves that, with probability at least $1 - cn^2e^{-\frac{1}{2}\gamma \log^2 n}$, we have

$$\phi(\hat{X}_t) \leq m(1 + c_1/n^2)^t \leq \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) e^{c_1 t}$$

for all $t \leq cn^2$.

Recall that $t_0 = c/8\lambda$. Thus, with probability at least $1 - e^{-cn^2/6}$, there are no more than cn^2 events in the continuous-time chain X during the interval $[0, t_0]$. Since ϕ is bounded above by C , it follows that, for all $t \leq t_0$,

$$\begin{aligned} \mathbb{E}[\phi(X_t)] &\leq \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) e^{8c_1 \lambda t_0} + C \left(8\lambda t_0 n^2 e^{-\frac{1}{2}\gamma \log^2 n} + e^{-\frac{\lambda t_0}{25} n^2} \right) \\ &\leq 2e^{8c_1 \lambda t_0} \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right). \end{aligned}$$

Substituting for the value of c_1 gives the required result, both in the general case and the case $d = 1$. \square

7. PROOF OF THEOREM 1.1

sec:proof-main

We now use the results of the previous section to derive the main theorem. We need one routine lemma, showing that the differential equation (1.3) is Lipschitz with an appropriate constant, in the domain of interest to us.

lem.lipschitz

Lemma 7.1. *Let d and C be positive integers. Let $\lambda > 0$. The differential equation (1.3) is Lipschitz with constant $8d^2(\lambda + 1)(C + 1)^2$ on the set $\{z \in \mathbb{R}^{C+1} : z(j) \geq 0 \text{ for each } j, \sum_j z(j) \leq 1\}$ with respect to the ℓ_∞ norm.*

For $d = 1$, the differential equation has Lipschitz constant $2\lambda + 2C + 6$ on $[0, 1]^{C+1}$, with respect to the ℓ_∞ norm.

Proof. For $0 < k < C$,

$$\begin{aligned} |F_k(x) - F_k(y)| &\leq \lambda|x(k-1) - y(k-1)| + \lambda|x(k) - y(k)| \\ &\quad + k|x(k) - y(k)| + (k+1)|x(k+1) - y(k+1)| \\ &\quad + |g_{k-1}(x) - g_{k-1}(y)| + |g_k(x) - g_k(y)|. \end{aligned}$$

Now, for $x, y \in \{z \in \mathbb{R}^{C+1} : z(j) \geq 0 \text{ for each } j, \sum_j z(j) \leq 1\}$,

$$\begin{aligned} & \left| 2x(C) \sum_{r=1}^d x(k)x(\leq k)(1 - x(\leq k)^2)^{r-1}(1 - x(\leq k-1)^2)^{d-r} \right. \\ & \left. - 2y(C) \sum_{r=1}^d y(k)y(\leq k)(1 - y(\leq k)^2)^{r-1}(1 - y(\leq k-1)^2)^{d-r} \right| \\ & \leq 2(2d + d(C+1) + d(d-1)(C+1)) \max_{0 \leq j \leq C} |x(j) - y(j)| \\ & \leq 3d^2(C+1) \max_{0 \leq j \leq C} |x(j) - y(j)|. \end{aligned}$$

Also,

$$\begin{aligned} & \left| x(C) \sum_{r=1}^d x(k) \sum_{i=k+1}^{C-1} x(i)(1 - x(\leq i)^2)^{r-1}(1 - x(\leq i-1)^2)^{d-r} \right. \\ & \left. - y(C) \sum_{r=1}^d y(k) \sum_{i=k+1}^{C-1} y(i)(1 - y(\leq i)^2)^{r-1}(1 - y(\leq i-1)^2)^{d-r} \right| \\ & \leq 3d^2C(C+1) \max_{0 \leq j \leq C} |x(j) - y(j)|. \end{aligned}$$

It follows that, for $k = 0, \dots, C-1$,

$$|g_k(x) - g_k(y)| \leq 3d^2(C+1)^2 \max_{0 \leq j \leq C} |x(j) - y(j)|. \quad (7.1)$$

So, for $0 < k < C$, for $x, y \in \{z \in \mathbb{R}^{C+1} : z(j) \geq 0 \text{ for each } j, \sum_j z(j) \leq 1\}$,

$$\begin{aligned} |F_k(x) - F_k(y)| &\leq (2\lambda + 2C + 6d^2(C+1)^2) \max_{0 \leq j \leq C} |x(j) - y(j)| \\ &\leq 8d^2(\lambda + 1)(C+1)^2 \max_{0 \leq j \leq C} |x(j) - y(j)|, \end{aligned}$$

and the same bound holds for $k = 0$ and $k = C$.

For $d = 1$, it is easy to see that, for $k = 0, \dots, C$ and $x, y \in [0, 1]^{C+1}$, $|g_k(x) - g_k(y)| \leq 6 \max_j |x(j) - y(j)|$, and therefore

$$|F_k(x) - F_k(y)| \leq (2\lambda + 2C + 6) \max_{0 \leq j \leq C} |x(j) - y(j)|.$$

for each $k = 0, \dots, C$. \square

lem.unique-sol

Lemma 7.2. *Let d and C be positive integers. Let $\lambda > 0$. Let ξ_0 satisfy $\sum_{j=0}^C \xi_0(i) = 1$ and $\xi_0(i) \geq 0$ for all i . The differential equation (1.3) has a unique solution (ξ_t) subject to initial condition ξ_0 , valid for all times $t \geq 0$. Furthermore, $\sum_{j=0}^C \xi_t(j) = 1$ and $\xi_t(j) \geq 0$ for all j , for all $t \geq 0$.*

Proof. By Lemma 7.1, F is locally Lipschitz with respect to the ℓ_∞ norm, so the differential equation (1.3) has a unique maximal solution (ξ_t) valid on $[0, t_{\max})$ for some $t_{\max} > 0$. Note that $\sum_{j=0}^C F_j(\xi) = 0$ for all ξ and so $\sum_{j=0}^C \xi_t(j)$ is constant for all times $t < t_{\max}$, and hence is equal to 1. Also, $F_j(\xi) \geq 0$ whenever $\xi(j) = 0$. By standard arguments, $\xi_t(j) \geq 0$ for all j and all $t < t_{\max}$. Thus $\xi_t \in \{z \in \mathbb{R}^{C+1} : z(j) \geq 0 \text{ for each } j, \sum_j |z(j)| \leq 1\}$ for all $t < t_{\max}$, and hence $t_{\max} = \infty$. \square

Lemma 7.3. *Let λ and t_0 be positive reals, let d and C be positive integers, and suppose that $n \geq n_0(\lambda, d, C, t_0)$. Let ξ_0 satisfy $\xi_0(j) \geq 0$ for all $j = 0, \dots, C$, and $\sum_{j=0}^C \xi_0(j) = 1$. Then, for each v and each $t \in [0, t_0]$,*

$$\begin{aligned} & \sup_j \left| \frac{1}{n-1} \mathbb{E}[f_{v,j}(X_t)] - \xi_t(j) \right| \\ & \leq \left(\sup_j \left| \frac{1}{n-1} f_{v,j}(x_0) - \xi_0(j) \right| \right. \\ & \quad \left. + 45\lambda t_0 d^2 (C+1)^3 \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) \right) e^{216(\lambda+1)d^2(C+1)^3 t_0}. \end{aligned}$$

For $d = 1$, we have

$$\begin{aligned} & \sup_j \left| \frac{1}{n-1} \mathbb{E}[f_{v,j}(X_t)] - \xi_t(j) \right| \\ & \leq \left(\sup_j \left| \frac{1}{n-1} f_{v,j}(x_0) - \xi_0(j) \right| \right. \\ & \quad \left. + 45\lambda t_0 (C+1)^3 \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) \right) e^{216(\lambda+1)(C+1)t_0}. \end{aligned}$$

Proof. For each j , for $t \leq t_0$, we have

$$\xi_t(j) = \xi_0(j) + \int_0^t F_j(\xi_u) du.$$

As before, for $v \in V_n$ and $j \in \{0, \dots, C\}$, $z_t(v, j) = (n-1)^{-1} \mathbb{E}[f_{v,j}(X_t)]$, and z_t^v is the vector $(z_t(v, j) : j \in \{0, \dots, C\})$. Also, A is the generator of the process X . Then, for every j ,

$$\begin{aligned} \frac{dz_t(v, j)}{dt} &= \frac{1}{n-1} \mathbb{E}(A f_{v,j}(X_t)) \\ &= \lambda z_t(v, j-1) - \lambda z_t(v, j) - j z_t(v, j) + (j+1) z_t(v, j+1) \\ &\quad + \frac{\lambda}{n-1} (\mathbb{E}[g_{v,j-1}(X_t)] - \mathbb{E}[g_{v,j}(X_t)]), \end{aligned}$$

so

$$\begin{aligned}
& z_t(v, j) - z_0(v, j) \\
&= \frac{1}{n-1} \int_0^t \mathbb{E}[Af_{v,j}(X_u)] du \\
&= \int_0^t \left(\lambda z_u(v, j-1) - \lambda z_u(v, j) - j z_u(v, j) + (j+1) z_u(v, j+1) \right) du \\
&\quad + \int_0^t \left(\frac{\lambda}{n-1} (\mathbb{E}[g_{v,j-1}(X_u)] - \mathbb{E}[g_{v,j}(X_u)]) \right) du \\
&= \int_0^t F_j(z_u^v) du + \lambda \int_0^t \left(\frac{1}{n-1} \mathbb{E}[g_{v,j-1}(X_u)] - g_{j-1}(z_u^v) \right) du \\
&\quad - \lambda \int_0^t \left(\frac{1}{n-1} \mathbb{E}[g_{v,j}(X_u)] - g_j(z_u^v) \right) du.
\end{aligned}$$

For each j and t , let $\epsilon_t(v, j) = \sup_{u \leq t} |z_u(v, j) - \xi_u(j)|$, and let ϵ_t^v be the vector with components $\epsilon_t(v, j)$ ($j = 0, \dots, C$). Let L be the Lipschitz constant of the function F , as in Lemma 7.1. Then we have, using Lemma 6.1,

$$\begin{aligned}
\|\epsilon_t^v\|_\infty &\leq \|\epsilon_0^v\|_\infty + \int_0^t \left(L \|\epsilon_u^v\|_\infty + 2\lambda \sup_{u \leq t} \max_j \left| \frac{1}{n-1} \mathbb{E}[g_{v,j}(X_u)] - g_j(z_u^v) \right| \right) du \\
&\leq \|\epsilon_0^v\|_\infty + L \int_0^t \|\epsilon_u^v\|_\infty du \\
&\quad + 45\lambda t d^2 (C+1)^3 \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) e^{208(\lambda+1)d^2(C+1)^3 t_0}.
\end{aligned}$$

By Gronwall's lemma, for each $t \leq t_0$,

$$\begin{aligned}
\|\epsilon_t^v\|_\infty &\leq e^{Lt_0} \left(\|\epsilon_0^v\|_\infty \right. \\
&\quad \left. + 45\lambda t_0 d^2 (C+1)^3 \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) e^{208(\lambda+1)d^2(C+1)^3 t_0} \right),
\end{aligned}$$

which gives the required result, since we may take $L = 8d^2(\lambda+1)(C+1)^2$ by Lemma 7.1. The result for $d = 1$ follows in an identical manner. \square

Proof of Theorem 1.1. Let

$$\begin{aligned}
w &= \left(\sup_{j,u} \left| \frac{1}{n-1} f_{u,j}(x_0) - \xi_0(j) \right| + 45\lambda t_0 d^2 (C+1)^3 \left(\phi(x_0) + \frac{3 \log n}{\sqrt{n}} \right) \right) \\
&\quad \times e^{216(\lambda+1)d^2(C+1)^3 t_0}.
\end{aligned}$$

The previous lemma, along with (5.19), yields

$$\begin{aligned}
&\mathbb{P} \left(\sup_{v,k,t} |f_{v,k}(X_t) - (n-1)\xi_t(k)| > (n-1)w + \frac{1}{2} n^{1/2} \log n \right) \\
&\leq \mathbb{P} \left(\sup_{v,k,t} |f_{v,k}(X_t) - \mathbb{E} f_{v,k}(X_t)| > \frac{1}{2} n^{1/2} \log n \right)
\end{aligned}$$

$$\leq 8cCn^3e^{-\gamma \log^2 n} \leq e^{-\frac{1}{2}\gamma \log^2 n},$$

where the supremum is over all $v \in V_n$, $k \in \{0, \dots, C\}$, and $t \leq t_0$. \square

The proof of Theorem 1.2 is essentially identical.

8. INITIAL CONDITIONS

sec:init-cond

Given a vertex v , in order for the functions $f_{v,k}(X_t)$ ($k = 0, \dots, C$) to be well approximated by the solution (ξ_t) to the differential equation (1.3), given the initial state X_0 of the system, we must choose initial condition ξ_0 for (1.3) in such a way that $\sup_{j \in \{0, \dots, C\}} \epsilon_0(v, j)$ is small, where $\epsilon_0(v, j) = |\xi_0(j) - (n-1)^{-1} \mathbb{E}[f_{v,j}(X_0)]|$. For instance, we can take $\xi_0(j) = (n-1)^{-1} \mathbb{E}[f_{v,j}(X_0)]$ for $j = 0, \dots, C$. In addition, there are restrictions on allowed initial states X_0 , to ensure that $\phi(X_0)$ is not too large.

Clearly, $X_0 = 0$ implies that $\phi^1(X_0) = 0$, so the law of large numbers in Theorem 1.1 holds if ξ_0 satisfies $\xi_0(0) = 1$ and $\xi_0(j) = 0$ for $j = 1, \dots, C$.

Now consider an initial state obtained as follows. For a constant $c_0 > 0$, we throw $\lfloor c_0 \binom{n}{2} \rfloor$ calls onto the network, one at a time. Each call chooses endpoints u and v uniformly at random; it is routed onto $\{u, v\}$ if there is spare capacity. Otherwise, it chooses an ordered list of d intermediate nodes (w_1, \dots, w_d) uniformly at random with replacement and is routed onto the first route $\{u, w_i\}, \{v, w_i\}$ minimising the maximum load of the two links, if this route has capacity. If each of the d routes has a full link, then the call is lost. Let X_0 be an initial state obtained from this $\lfloor c_0 \binom{n}{2} \rfloor$ -step allocation. We will show that, with high probability, $\phi(X_0)$ is at most $3n^{-1/2} \log n$.

We start by analysing $\phi^1(X_0)$, using the bound

$$\begin{aligned} \phi^1(X_0) &= \max_{u,v:u \neq v} \max_{j,k} \left| \frac{1}{n-2} \sum_w \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right. \\ &\quad \left. - \frac{1}{(n-2)^2} \sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0) \right| \\ &\leq \max_{u,v:u \neq v} \max_{j,k} \frac{1}{n-2} \left| \sum_w \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) - \mathbb{E} \left[\sum_w \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right] \right| \\ &\quad + \max_{u,v:u \neq v} \max_{j,k} \frac{1}{n-2} \left| \mathbb{E} \left[\sum_w \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right] \right. \\ &\quad \left. - \frac{1}{n-2} \mathbb{E} \left[\sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0) \right] \right| \\ &\quad + \max_{u,v:u \neq v} \max_{j,k} \frac{1}{(n-2)^2} \left| \mathbb{E} \left[\sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0) \right] \right. \\ &\quad \left. - \sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0) \right| \end{aligned}$$

Arguments similar to those in Sections 4 and 5 show that $\sum_w \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0)$ and $\sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0)$ are well-concentrated. Specifically, there exists a constant $\gamma_0 > 0$ such that for all u, v, j, k ,

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) - \mathbb{E} \left[\sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right] \right| \geq \frac{1}{2} \sqrt{n} \log n \right) \\ \leq 8e^{-\gamma_0 \log^2 n}, \end{aligned}$$

$$\begin{aligned} \mathbb{P} \left(\frac{1}{n-2} \left| \sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0) - \mathbb{E} \left[\sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0) \right] \right| \right. \\ \left. \geq \frac{1}{2} \sqrt{n} \log n \right) \leq 8e^{-\gamma_0 \log^2 n}, \end{aligned}$$

and also, for all u, j ,

$$\mathbb{P} \left(|f_{u,j}(X_0) - \mathbb{E}[f_{u,j}(X_0)]| \geq \frac{1}{2} \sqrt{n} \log n \right) \leq 8e^{-\gamma_0 \log^2 n}. \quad (8.1) \quad \boxed{\text{eq. dev}}$$

We deduce that, with probability at least $1 - 16(C+1)^2 n^2 e^{-\gamma_0 \log^2 n}$,

$$\begin{aligned} \phi^1(X_0) &\leq \max_{u,v:u \neq v} \max_{j,k} \frac{1}{n-2} \left| \mathbb{E} \left[\sum_w \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right] \right. \\ &\quad \left. - \frac{1}{n-2} \mathbb{E} \left[\sum_{w \neq u,v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k(X_0) \right] \right| + 2 \frac{\log n}{\sqrt{n}}. \end{aligned}$$

From (8.1) we have that, for sufficiently large n ,

$$\begin{aligned} &\mathbb{E} [f_{u,j}(X_0) f_{u,k}(X_0)] - \mathbb{E} f_{u,j}(X_0) \mathbb{E} f_{u,k}(X_0) \\ &= \mathbb{E} [(f_{u,j}(X_0) - \mathbb{E} f_{u,j}(X_0)) (f_{u,k}(X_0) - \mathbb{E} f_{u,k}(X_0))] \\ &\leq \left(\frac{1}{2} \sqrt{n} \log n \right)^2 + 16e^{-\gamma_0 \log^2 n} n^2 \leq \frac{1}{2} n \log^2 n. \end{aligned}$$

Now we use the fact that, for fixed w , all of the variables $X_0(\{u, w\})$ are identically distributed, to obtain that

$$\mathbb{E} f_{w,j}(X_0) f_{w,k}(X_0) = (n-1) \mathbb{E} \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{uw}^k(X_0) + (n-1)(n-2) \mathbb{E} \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0)$$

for any distinct u and v , and therefore

$$\left| \frac{1}{(n-1)^2} \mathbb{E} f_{w,j}(X_0) f_{w,k}(X_0) - \mathbb{E} \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right| \leq \frac{1}{n-1}.$$

Note also that $\mathbb{E} f_{w,j}(X_0) = (n-1) \mathbb{E} \mathbb{I}_{uw}^j(X_0)$, for any u and w . Hence

$$\begin{aligned} &\left| \mathbb{E} \left[\mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right] - \mathbb{E} \mathbb{I}_{uw}^j(X_0) \mathbb{E} \mathbb{I}_{vw}^k(X_0) \right| \\ &\leq \frac{1}{n-1} + \frac{1}{(n-1)^2} \frac{1}{2} n \log^2 n \leq \frac{\log^2 n}{n}, \end{aligned} \quad (8.2)$$

for any distinct u, v, w , and any j and k . The same argument gives

$$\left| \mathbb{E} \left[\mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw'}^k(X_0) \right] - \mathbb{E} \mathbb{I}_{uw}^j(X_0) \mathbb{E} \mathbb{I}_{vw'}^k(X_0) \right| \leq \frac{\log^2 n}{n} \quad (8.3) \quad \boxed{\text{eq. indep-4}}$$

whenever u, v, w and w' are all distinct, and for any j and k .

Now, by (8.2) and (8.3), for any distinct u and v , and any j and k ,

$$\begin{aligned} & \frac{1}{n-2} \left| \mathbb{E} \left[\sum_{w \neq u, v} \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right] - \frac{1}{n-2} \mathbb{E} \left[\sum_{w \neq u, v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u, v} \mathbb{I}_{vw'}^k(X_0) \right] \right| \\ & \leq \frac{2 \log^2 n}{n} + \frac{1}{n-2} \left| \sum_{w \neq u, v} \mathbb{E}[\mathbb{I}_{uw}^j(X_0)] \mathbb{E}[\mathbb{I}_{vw}^k(X_0)] \right. \\ & \quad \left. - \frac{1}{n-2} \sum_{w \neq u, v} \mathbb{E}[\mathbb{I}_{uw}^j(X_0)] \sum_{w' \neq u, v} \mathbb{E}[\mathbb{I}_{vw'}^k(X_0)] \right|. \end{aligned}$$

Since all the \mathbb{I}_{uw}^j are identically distributed, as are all the \mathbb{I}_{vw}^k , this last term is zero, and we have, for any distinct u, v , and any j, k ,

$$\begin{aligned} & \frac{1}{n-2} \left| \mathbb{E} \left[\sum_{w \neq u, v} \mathbb{I}_{uw}^j(X_0) \mathbb{I}_{vw}^k(X_0) \right] - \frac{1}{n-2} \mathbb{E} \left[\sum_{w \neq u, v} \mathbb{I}_{uw}^j(X_0) \sum_{w' \neq u, v} \mathbb{I}_{vw'}^k(X_0) \right] \right| \\ & \leq 2 \frac{\log^2 n}{n}. \end{aligned}$$

It follows that, for n large enough,

$$\mathbb{P}(\phi^1(X_0) \geq 3 \frac{\log n}{\sqrt{n}}) \leq 16(C+1)^2 n^2 e^{-\gamma_0 \log^2 n}.$$

Furthermore,

$$\begin{aligned} \phi^2(X_0) &= \max_{u, v: u \neq v} \max_j \frac{1}{n-2} |f_{u,j}(X_0) - f_{v,j}(X_0)| \\ &\leq \max_{u, v: u \neq v} \max_j \frac{1}{n-2} \left(|f_{u,j}(X_0) - \mathbb{E}[f_{u,j}(X_0)]| \right. \\ & \quad \left. + |f_{v,j}(X_0) - \mathbb{E}[f_{v,j}(X_0)]| \right), \end{aligned}$$

and hence, by (8.1),

$$\mathbb{P}(\phi^2(X_0) \geq 2 \frac{\log n}{\sqrt{n}}) \leq 8(C+1)n^2 e^{-\gamma_0 \log^2 n}.$$

For ϕ^3 , standard Poisson tail bounds yield that, for each fixed pair $\{u, v\}$, the probability that there are more than $c_0 \log^2 n$ calls with endpoints u and v is at most $(e/\log^2 n)^{c_0 \log^2 n} \leq e^{-\gamma_0 \log^2 n}$ for sufficiently large n . Thus

$$\mathbb{P} \left(\phi^3(X_0) > \frac{c_0 \log^2 n}{n} \right) \leq n^2 e^{-\gamma_0 \log^2 n}.$$

Hence, as claimed, for n large enough,

$$\mathbb{P}(\phi(X_0) \geq 3 \frac{\log n}{\sqrt{n}}) \leq 25(C+1)^2 n^2 e^{-\gamma_0 \log^2 n} \leq e^{-\frac{1}{2}\gamma_0 \log^2 n}.$$

9. EXTENSIONS

Theorems 1.1 and 1.2 imply a ‘global’ law of large numbers approximation for the network, that is the number $f_k(X_t)$ of links with load k is well approximated by the differential equation (1.3). Indeed, for instance, by Theorem 1.1, when $d \geq 2$, summing over all the vertices gives the following. Let A_n be the event that, for each k and each $t \in [0, t_0]$,

$$\begin{aligned} |f_k(X_t) - \binom{n}{2} \xi_t(k)| &\leq \left(\sup_j |f_j(X_0) - \binom{n}{2} \xi_0(j)| \right. \\ &\quad \left. + 23(\lambda+1)(t_0+1)d^2(C+1)^3 \left(n^2 \phi(X_0) + 3n^{3/2} \log n \right) \right) e^{216(\lambda+1)d^2(C+1)^3 t_0}. \end{aligned}$$

Then $\mathbb{P}(\overline{A_n}) \leq e^{-\frac{1}{2}\gamma \log^2 n}$. In the case $d = 1$, an analogous result can be deduced from Theorem 1.2. It would appear that these results are unlikely to be close to best possible: we would expect to be able to approximate $f_k(X_t)$ with error of order $O(n)$, up to a logarithmic term, but have not been able to prove such a result using our methods. There are several places where our argument would need to be improved, including the concentration of measure arguments used in the proofs of Lemma 6.4 and of Lemma 6.2.

Our techniques can be adapted to analyse all the other variants of the model mentioned in the introduction. More generally, one would expect to be able to handle models involving a large system (of size n), where any pair of elements (e.g. links) interact at a rate tending to 0 as $n \rightarrow \infty$. (In the present model, any pair of links share an arrival stream at a rate of order $O(1/n)$.) These extensions may require a modified definition of function ϕ .

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APPENDIX A. PROOF OF LEMMA 6.3

SApp

Proof of Lemma 6.3. We start with $\phi_{u,v,j,k}^1$, and note that

$$\Delta\phi_{u,v,j,k}^1 = \frac{1}{n-2}\Delta\left(\sum_{w \neq u,v} \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k\right) - \frac{1}{(n-2)^2}\Delta\left(\sum_{w \neq u,v} \mathbb{I}_{uw}^j \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k\right).$$

We therefore need to compute and compare the conditional expectations of these two increments.

On the event $B_{t-1} = \{\hat{X}_s \in \tilde{S} \text{ for all } s \leq t-1\}$, we have, for any u, v, j, k ,

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor\right) \mathbb{E}\left[\Delta\left(\sum_{w \neq u,v} \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k\right)(\hat{X}_t) \mid \mathcal{F}_{t-1}\right] \\ &= \left\{ \sum_{w \neq u,v} \mathbb{I}_{vw}^k \left[-j \mathbb{I}_{uw}^j + (j+1) \mathbb{I}_{uw}^{j+1} + \lambda(\mathbb{I}_{uw}^{j-1} - \mathbb{I}_{uw}^j + g_{u,w,j-1} - g_{u,w,j}) \right] \right. \\ & \quad + \sum_{w \neq u,v} \mathbb{I}_{uw}^j \left[-k \mathbb{I}_{vw}^k + (k+1) \mathbb{I}_{vw}^{k+1} + \lambda(\mathbb{I}_{vw}^{k-1} - \mathbb{I}_{vw}^k + g_{v,w,k-1} - g_{v,w,k}) \right] \\ & \quad + \lambda \sum_{w \neq u,v} (P_{u,v,w,j-1,k-1} + P_{u,v,w,j,k}) \Big\} (\hat{X}_{t-1}) \\ & \quad + \sum_{w \neq u,v} \hat{X}_{t-1}(\{u,v\}, w) (\mathbb{I}_{uw}^j \mathbb{I}_{vw}^k + \mathbb{I}_{uw}^{j+1} \mathbb{I}_{vw}^{k+1}) (\hat{X}_{t-1}), \end{aligned} \tag{A.1}$$

where, for instance, $\lambda(\mathbb{I}_{vw}^k g_{u,w,j})(\hat{X}_{t-1})$ is the contribution for the case where a call is indirectly routed via the link uw which has load j in \hat{X}_{t-1} , and $\lambda P_{u,v,w,j-1,k-1}(\hat{X}_{t-1})$ is the contribution for arrivals onto the route consisting of the links $\{u, w\}$ and $\{v, w\}$, with loads $j-1$ and $k-1$ respectively in \hat{X}_{t-1} . The term $\sum_{w \neq u,v} \hat{X}_{t-1}(\{u,v\}, w) \mathbb{I}_{uw}^{j+1} \mathbb{I}_{vw}^{k+1}(\hat{X}_{t-1})$ represents departures of calls from the route consisting of links $\{u, w\}$ and $\{v, w\}$, with loads $j+1$ and $k+1$ respectively, while the term $\sum_{w \neq u,v} \hat{X}_{t-1}(\{u,v\}, w) \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k(\hat{X}_{t-1})$ represents departures of calls from the route consisting of links $\{u, w\}$ and

$\{v, w\}$, with loads j and k respectively, which, along with the arrivals to such routes, have otherwise been overcounted.

Explicitly, in the expression above, for each u, w , and j ,

$$g_{u,w,j} = P_{u,w,j}^+ + P_{u,w,j}^- + Q_{u,w,j}^+ + Q_{u,w,j}^-,$$

with

$$P_{u,w,j}^+ = \frac{1}{(n-2)^d} \sum_{r=1}^d \mathbb{I}_{uw}^j \sum_{v', \mathbf{w}_r} \mathbb{I}_{v'u}^C \mathbb{I}_{v'w}^{\leq j} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{v'u, w_s}^{\leq j}) \prod_{s=r+1}^d (1 - \mathbb{I}_{v'u, w_s}^{\leq j-1});$$

$$P_{u,w,j}^- = \frac{1}{(n-2)^d} \sum_{r=1}^d \mathbb{I}_{uw}^j \sum_{v', \mathbf{w}_r} \mathbb{I}_{v'u}^C \sum_{i=j+1}^{C-1} \mathbb{I}_{v'w}^i \prod_{s=1}^{r-1} (1 - \mathbb{I}_{v'u, w_s}^{\leq i}) \prod_{s=r+1}^d (1 - \mathbb{I}_{v'u, w_s}^{\leq i-1}),$$

$$Q_{u,w,j}^+ = \frac{1}{(n-2)^d} \sum_{r=1}^d \mathbb{I}_{uw}^j \sum_{v', \mathbf{w}_r} \mathbb{I}_{v'w}^C \mathbb{I}_{v'u}^{\leq j} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{v'w, w_s}^{\leq j}) \prod_{s=r+1}^d (1 - \mathbb{I}_{v'w, w_s}^{\leq j-1});$$

$$Q_{u,w,j}^- = \frac{1}{(n-2)^d} \sum_{r=1}^d \mathbb{I}_{uw}^j \sum_{v', \mathbf{w}_r} \mathbb{I}_{v'w}^C \sum_{i=j+1}^{C-1} \mathbb{I}_{v'u}^i \prod_{s=1}^{r-1} (1 - \mathbb{I}_{v'w, w_s}^{\leq i}) \prod_{s=r+1}^d (1 - \mathbb{I}_{v'w, w_s}^{\leq i-1}),$$

where, according to our convention, $\sum_{v'}$ denotes the sum over all $v' \neq u, w$; in the first two expressions, $\sum_{\mathbf{w}_r}$ denotes the sum over all $w_1, \dots, w_{r-1}, w_{r+1}, \dots, w_d$ such that $w_s \neq v', u$ for any s , whereas, in the final two expressions, $\sum_{\mathbf{w}_r}$ denotes the sum over all $w_1, \dots, w_{r-1}, w_{r+1}, \dots, w_d$ such that $w_s \neq v', w$ for any s . Also, explicitly,

$$P_{u,v,w,j,k} = \frac{1}{(n-2)^d} \sum_{r=1}^d \mathbb{I}_{uv}^C \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k \sum_{\mathbf{w}_r} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv, w_s}^{\leq j \vee k}) \prod_{s=r+1}^d (1 - \mathbb{I}_{uv, w_s}^{\leq (j \vee k)-1}),$$

where here $\sum_{\mathbf{w}_r}$ denotes the sum over all $w_1, \dots, w_{r-1}, w_{r+1}, \dots, w_d$ such that $w_s \neq u, v$ for any s .

Similarly, on the event $B_{t-1} = \{\hat{X}_s \in \tilde{S} \text{ for all } s \leq t-1\}$,

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} \left[\Delta \left(\sum_{w \neq u, v} \mathbb{I}_{uw}^j \sum_{w' \neq u, v} \mathbb{I}_{vw'}^k \right) (\hat{X}_t) \mid \hat{\mathcal{F}}_{t-1} \right] \\ &= \left\{ \left(\sum_{w' \neq u, v} \mathbb{I}_{vw'}^k \right) \sum_{w \neq u, v} \left[-j \mathbb{I}_{uw}^j + (j+1) \mathbb{I}_{uw}^{j+1} + \lambda (\mathbb{I}_{uw}^{j-1} - \mathbb{I}_{uw}^j) \right. \right. \\ & \quad \left. \left. + g_{u,w,j-1} - g_{u,w,j} \right] + \left(\sum_{w \neq u, v} \mathbb{I}_{uw}^j \right) \sum_{w' \neq u, v} \left[-k \mathbb{I}_{vw'}^k + (k+1) \mathbb{I}_{vw'}^{k+1} \right. \right. \\ & \quad \left. \left. + \lambda (\mathbb{I}_{vw'}^{k-1} - \mathbb{I}_{vw'}^k + g_{v,w',k-1} - g_{v,w',k}) \right] \right. \\ & \quad \left. + \lambda \sum_{w \neq u, v} (P_{u,v,w,j-1,k-1} + P_{u,v,w,j,k}) \right\} (\hat{X}_{t-1}) \end{aligned}$$

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$$+ \sum_{w \neq u,v} \hat{X}_{t-1}(\{u, v\}, w) (\mathbb{I}_{uw}^j \mathbb{I}_{vw}^k + \mathbb{I}_{uw}^{j+1} \mathbb{I}_{vw}^{k+1}) (\hat{X}_{t-1}). \quad (\text{A.2})$$

Comparing corresponding terms in the two expressions (A.1) and (A.2) gives that, on the event B_{t-1} ,

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} [|\Delta \phi_{u,v,j,k}^1(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \\ & \leq (2j + 2k + 2 + 4\lambda) \phi^1(\hat{X}_{t-1}) \\ & \quad + \lambda \frac{n-3}{(n-2)^2} \sum_{w \neq u,v} (P_{u,v,w,j-1,k-1} + P_{u,v,w,j,k})(\hat{X}_{t-1}) \\ & \quad + \frac{n-3}{(n-2)^2} \sum_{w \neq u,v} \hat{X}_{t-1}(\{u, v\}, w) (\mathbb{I}_{uw}^j \mathbb{I}_{vw}^k + \mathbb{I}_{uw}^{j+1} \mathbb{I}_{vw}^{k+1}) (\hat{X}_{t-1}) \\ & \quad + |a_{u,v,j,k-1} - a_{u,v,j,k} + a_{v,u,k,j-1} - a_{v,u,k,j}|(\hat{X}_{t-1}), \end{aligned} \quad (\text{A.3})$$

where

$$\begin{aligned} a_{u,v,j,k} &= \frac{\lambda}{n-2} \sum_w g_{u,w,j} \left(\mathbb{I}_{vw}^k - \left(\frac{1}{n-2} \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k \right) \right) \\ &= \frac{\lambda}{n-2} \sum_w (P_{u,w,j}^+ + P_{u,w,j}^- + Q_{u,w,j}^+ + Q_{u,w,j}^-) \\ & \quad \times \left(\mathbb{I}_{vw}^k - \left(\frac{1}{n-2} \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k \right) \right). \end{aligned}$$

Bounding the middle two terms in the expression (A.3) above is straightforward: note that each $P_{u,v,w,j,k}$ is at most $d/(n-2)$, while

$$\sum_{w \neq u,v} \hat{X}_{t-1}(\{u, v\}, w) \leq (n-2) \phi^3(\hat{X}_{t-1}),$$

and so, on B_{t-1} ,

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} [|\Delta \phi_{u,v,j,k}^1(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \\ & \leq (4C + 2 + 4\lambda) \phi^1(\hat{X}_{t-1}) + \frac{2d\lambda}{n-2} + 2\phi^3(\hat{X}_{t-1}) \\ & \quad + |a_{u,v,j,k-1} - a_{u,v,j,k} + a_{v,u,k,j-1} - a_{v,u,k,j}|(\hat{X}_{t-1}) \\ & \leq (4C + 4 + 4\lambda) \phi(\hat{X}_{t-1}) + \frac{2d\lambda}{n-2} \\ & \quad + |a_{u,v,j,k-1} - a_{u,v,j,k} + a_{v,u,k,j-1} - a_{v,u,k,j}|(\hat{X}_{t-1}). \end{aligned}$$

We will now show that both

$$\frac{1}{n-2} \sum_{w \neq u,v} P_{u,w,j}^+ \left(\frac{1}{n-2} \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k \right) \quad \text{and} \quad \frac{1}{n-2} \sum_{w \neq u,v} P_{u,w,j}^+ \mathbb{I}_{vw}^k$$

are close to their ‘standardised’ version

$$\begin{aligned} \hat{P}_{u,v,j,k}^+ &= \frac{1}{(n-2)^{2d+2}} \sum_{r=1}^d \sum_w \mathbb{I}_{uw}^j \sum_{w'} \mathbb{I}_{vw'}^k \sum_{v'} \mathbb{I}_{v'u}^C \sum_{v''} \mathbb{I}_{v''w}^{\leq j} \\ &\quad \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}). \end{aligned}$$

Analogous bounds hold if $P_{u,w,j}^+$ is replaced by $P_{u,w,j}^-$, $Q_{u,w,j}^+$ or $Q_{u,w,j}^-$.

First, an elementary calculation similar to earlier ones shows that

$$\begin{aligned} &\left| \frac{1}{n-2} \sum_{w \neq u,v} P_{u,w,j}^+ \mathbb{I}_{vw}^k - \hat{P}_{u,v,j,k}^+ \right| \\ &\leq d(d-1)(C+1)^2 \phi^1 + \left| \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_w \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k \sum_{v'} \mathbb{I}_{v'u}^C \mathbb{I}_{v'w}^{\leq j} \right. \\ &\quad \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{uw_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}) - \hat{P}_{u,v,j,k}^+ \left. \right| \\ &\leq d(d-1)(C+1)^2 \left(\phi^1 + \phi^2 + \frac{2}{n-2} \right) \\ &\quad + \sum_{r=1}^d \left(1 - \left(\frac{f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j}}{n-2} \right)^2 \right)^{r-1} \left(1 - \left(\frac{f_{u,\leq j-1} - \mathbb{I}_{uv}^{\leq j-1}}{n-2} \right)^2 \right)^{d-r} \\ &\quad \times \left| \frac{1}{(n-2)^2} \sum_w \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k \sum_{v' \neq u,w} \mathbb{I}_{v'u}^C \mathbb{I}_{v'w}^{\leq j} \right. \\ &\quad \left. - \frac{1}{(n-2)^4} \sum_{w \neq u,v} \mathbb{I}_{uw}^j \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k \sum_{v' \neq u,w} \mathbb{I}_{v'u}^C \sum_{v'' \neq u,w} \mathbb{I}_{v''w}^{\leq j} \right|, \end{aligned}$$

and similarly that

$$\begin{aligned} &\left| \frac{1}{(n-2)^2} \sum_{w \neq u,v} \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k \sum_{v' \neq u,w} \mathbb{I}_{v'u}^C \mathbb{I}_{v'w}^{\leq j} \right. \\ &\quad \left. - \frac{1}{(n-2)^4} \sum_{w \neq u,v} \mathbb{I}_{uw}^j \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k \sum_{v' \neq u,w} \mathbb{I}_{v'u}^C \sum_{v'' \neq u,w} \mathbb{I}_{v''w}^{\leq j} \right| \\ &\leq (C+1) \phi^1 + \left| \frac{1}{(n-2)^3} \sum_{w \neq u,v} \mathbb{I}_{uw}^j \mathbb{I}_{vw}^k \sum_{v' \neq u,w} \mathbb{I}_{v'u}^C \sum_{v'' \neq u,w} \mathbb{I}_{v''w}^{\leq j} \right. \\ &\quad \left. - \frac{1}{(n-2)^4} \sum_{w \neq u,v} \mathbb{I}_{uw}^j \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k \sum_{v' \neq u,w} \mathbb{I}_{v'u}^C \sum_{v'' \neq u,w} \mathbb{I}_{v''w}^{\leq j} \right| \\ &\leq (C+1) \left(\phi^1 + \phi^2 + \frac{2}{n-2} + \frac{f_{u,C} - \mathbb{I}_{uv}^C}{n-2} \frac{f_{u,\leq j} - \mathbb{I}_{uv}^{\leq j}}{n-2} \phi^1 \right) \\ &\leq (C+1) \left(2\phi^1 + \phi^2 + \frac{2}{n-2} \right). \end{aligned}$$

It follows that

$$\begin{aligned} \left| \frac{1}{n-2} \sum_w P_{u,w,j}^+ \mathbb{I}_{vw}^k - \hat{P}_{u,v,j,k}^+ \right| &\leq d(d-1)(C+1)^2 \left(\phi^1 + \phi^2 + \frac{2}{n-2} \right) \\ &\quad + d(C+1) \left(2\phi^1 + \phi^2 + \frac{2}{n-2} \right) \\ &\leq d^2(C+1)^2 \left(2\phi^1 + \phi^2 + \frac{2}{n-2} \right). \end{aligned}$$

If $d = 1$, we may replace the above bound by $(C+1)(2\phi^1 + \phi^2 + \frac{2}{n-2})$.

Similarly, but slightly more easily,

$$\left| \frac{1}{(n-2)^2} \sum_w P_{u,w,j}^+ \sum_{w' \neq u,v} \mathbb{I}_{vw'}^k - \hat{P}_{u,v,j,k}^+ \right| \leq d^2(C+1)^2 \left(2\phi^1 + \phi^2 + \frac{2}{n-2} \right).$$

Hence

$$\left| \frac{1}{n-2} \sum_w P_{u,w,j}^+ \left(\mathbb{I}_{vw}^k - \left(\frac{1}{n-2} \sum_{w'} \mathbb{I}_{vw'}^k \right) \right) \right| \leq 2d^2(C+1)^2 \left(2\phi^1 + \phi^2 + \frac{2}{n-2} \right).$$

Similar calculations for $P_{u,w,j}^-$, $Q_{u,w,j}^+$, $Q_{u,w,j}^-$ show that, for each u, v, j, k ,

$$|a_{u,v,j,k}| \leq 4\lambda d^2(C+1)^3 \left(2\phi^1 + \phi^2 + \frac{2}{n-2} \right).$$

If $d = 1$, we have the improved bound $|a_{u,v,j,k}| \leq 4\lambda(C+1)(2\phi^1 + \phi^2 + \frac{2}{n-2})$.

Hence, on B_{t-1} , we have, for all u, v, k, j ,

$$\begin{aligned} &\left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} [|\Delta \phi_{u,v,j,k}^1(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \\ &\leq (4C + 4 + 4\lambda) \phi(\hat{X}_{t-1}) + \frac{14d\lambda}{n-2} + 16\lambda d^2(C+1)^3 \left(3\phi(\hat{X}_{t-1}) + \frac{2}{n-2} \right) \\ &\leq 52(\lambda + 1)d^2(C+1)^3 \phi(\hat{X}_{t-1}) + \frac{46\lambda d^2(C+1)^3}{n-2}. \end{aligned}$$

For $d = 1$, we have the improved bound

$$\begin{aligned} &\left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} [|\Delta \phi_{u,v,j,k}^1(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \\ &\leq 52(\lambda + 1)(C+1) \phi(\hat{X}_{t-1}) + \frac{46\lambda(C+1)}{n-2}. \end{aligned}$$

Next, we consider $\Delta \phi_{u,v,j}^2 = \frac{1}{n-2} \Delta(f_{u,j} - f_{v,j})$, for $u, v \in V_n$ and $0 < j < C$. We have

$$\begin{aligned} &\left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} [|\Delta(f_{u,j}(\hat{X}_t) - f_{v,j}(\hat{X}_t))| \mid \hat{\mathcal{F}}_{t-1}] \\ &\leq \left\{ \lambda |f_{u,j-1} - f_{v,j-1}| + (\lambda + j) |f_{u,j} - f_{v,j}| + (j+1) |f_{u,j+1} - f_{v,j+1}| \right. \\ &\quad \left. + \lambda |g_{u,j-1} - g_{v,j-1}| + \lambda |g_{u,j} - g_{v,j}| \right\} (\hat{X}_{t-1}). \end{aligned}$$

We now refer to the functions $\hat{P}_{u,j}^+$, $\hat{P}_{u,j}^-$, $\hat{Q}_{u,j}^+$ and $\hat{Q}_{u,j}^-$ defined in the proof of Lemma 6.2, as well as their sum $\hat{g}_{u,j}$. Using (6.1), we have that

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E}[|\Delta(f_{u,j}(\hat{X}_t) - f_{v,j}(\hat{X}_t))| \mid \hat{\mathcal{F}}_{t-1}] \\ & \leq \left\{ (2\lambda + 2C + 1)(n-2)\phi^2 + 24d^2(C+1)^3(n-2)\phi \right. \\ & \quad \left. + \lambda|\hat{g}_{u,j-1} - \hat{g}_{v,j-1}| + \lambda|\hat{g}_{u,j} - \hat{g}_{v,j}| \right\} (\hat{X}_{t-1}). \end{aligned}$$

If $d = 1$, the term $24d^2(C+1)^3(n-2)\phi$ becomes $16(C+1)(n-2)\phi$, by (6.2).

An easy calculation shows that, for $n \geq 4$, and each u, v and j ,

$$\begin{aligned} & |\hat{P}_{u,j}^+ - \hat{P}_{v,j}^+| \\ &= \left| \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_{u'} \mathbb{I}_{u'u}^C \sum_{w_r, w'_r} \mathbb{I}_{uw_r}^j \mathbb{I}_{u'w'_r}^{\leq j} \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j} \mathbb{I}_{uw'_s}^{\leq j}) \right. \\ & \quad \times \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j-1} \mathbb{I}_{uw'_s}^{\leq j-1}) - \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_{u'} \mathbb{I}_{u'v}^C \sum_{w_r, w'_r} \mathbb{I}_{vw_r}^j \mathbb{I}_{u'w'_r}^{\leq j} \\ & \quad \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}) \Big| \\ & \leq d^2 C \left((n-2)\phi^2 + 2 \right) + \left| \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_{u'} \mathbb{I}_{u'u}^C \sum_{w_r, w'_r} \mathbb{I}_{vw_r}^j \mathbb{I}_{u'w'_r}^{\leq j} \right. \\ & \quad \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}) \\ & \quad - \frac{1}{(n-2)^{2d}} \sum_{r=1}^d \sum_{u'} \mathbb{I}_{u'v}^C \sum_{w_r, w'_r} \mathbb{I}_{vw_r}^j \mathbb{I}_{u'w'_r}^{\leq j} \\ & \quad \times \prod_{s=1}^{r-1} \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j} \mathbb{I}_{vw'_s}^{\leq j}) \prod_{s=r+1}^d \sum_{w_s, w'_s} (1 - \mathbb{I}_{u'w_s}^{\leq j-1} \mathbb{I}_{vw'_s}^{\leq j-1}) \Big| \\ & \leq d^2(C+1) \left((n-2)\phi^2 + 2 \right) + d(2d+1)(C+1) \left((n-2)\phi^2 + 2 \right) \\ & \leq 4d^2(C+1) \left((n-2)\phi^2 + 2 \right). \end{aligned}$$

Similarly, for each u, v and j ,

$$|\hat{P}_{u,j}^- - \hat{P}_{v,j}^-| \leq 4d^2(C+1)^2 \left((n-2)\phi^2 + 2 \right);$$

$$|\hat{Q}_{u,j}^+ - \hat{Q}_{v,j}^+| \leq 4d^2(C+1) \left((n-2)\phi^2 + 2 \right);$$

$$|\hat{Q}_{u,j}^- - \hat{Q}_{v,j}^-| \leq 4d^2(C+1)^2((n-2)\phi^2 + 2).$$

It follows that for $n \geq 4$, and each u, v and j , on B_{t-1} ,

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E}[|\Delta\phi_{u,v,j}^2(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \\ & \leq (2\lambda + 2C + 1 + 24d^2(C+1)^3 + 32\lambda d^2(C+1)^2)\phi(\hat{X}_{t-1}) \\ & \quad + \frac{64}{n-2}\lambda d^2(C+1)^2 \\ & \leq 35(\lambda+1)d^2(C+1)^3\phi(\hat{X}_{t-1}) + \frac{64\lambda d^2(C+1)^2}{n-2}. \end{aligned}$$

In the case $d = 1$, we may obtain

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E}[|\Delta\phi_{u,v,j}^2(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \\ & \leq 35(\lambda+1)(C+1)\phi(\hat{X}_{t-1}) + \frac{64\lambda(C+1)}{n-2}. \end{aligned}$$

Finally, we consider the expectation of the absolute value of $\Delta\phi_{u,v}^3(\hat{X}_t)$, conditional on $\hat{\mathcal{F}}_{t-1}$. We have

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} \left[\Delta\phi_{u,v}^3(\hat{X}_t) \mid \hat{\mathcal{F}}_{t-1} \right] \\ & = \left(\frac{\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor}{n-2} \right) \mathbb{E} \left[\Delta \left(\sum_{w \neq u,v} \hat{X}_t(\{u, v\}, w) \right) \mid \hat{\mathcal{F}}_{t-1} \right] \\ & = -\frac{1}{n-2} \sum_{w \neq u,v} \hat{X}_{t-1}(\{u, v\}, w) + \frac{\lambda \mathbb{I}_{uv}^C}{n-2} \left(1 - \frac{1}{(n-2)^d} \sum_{\mathbf{w}} \prod_{s=1}^d (1 - \mathbb{I}_{uv, w_s}^{\leq C-1}) \right), \end{aligned}$$

on event B_{t-1} . So we have

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E} \left[|\Delta\phi_{u,v}^3(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1} \right] \\ & \leq \frac{1}{n-2} \left(\sum_{w \neq u,v} \hat{X}_{t-1}(\{u, v\}, w) + \lambda \mathbb{I}_{uv}^C \right) \leq \phi^3(\hat{X}_{t-1}) + \frac{\lambda}{n-2}. \end{aligned}$$

For ρ any of the functions under consideration, we now have

$$\begin{aligned} & \left(\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \right) \mathbb{E}[|\Delta\rho(\hat{X}_t)| \mid \hat{\mathcal{F}}_{t-1}] \\ & \leq 52(\lambda+1)d^2(C+1)^3\phi(\hat{X}_{t-1}) + \frac{64\lambda d^2(C+1)^3}{n-2}. \end{aligned}$$

For $n \geq n_0$, $\lambda \binom{n}{2} + \lfloor 6\lambda \binom{n}{2} \rfloor \geq 6\lambda \binom{n}{2} \geq 2n^2$, and the lemma follows. \square